SOME STRUCTURE THEOREMS FOR TOPOLOGICAL MACHINES

By

EUGENE MICHAEL NORRIS

A DISSERTATION PRESENTED TO THE GRADUATE COUNCIL OF THE UNIVERSITY OF FLORIDA IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

UNIVERSITY OF FLORIDA

1969

TABLE OF CONTENTS

INTRODUCTION

This dissertation is not an introduction to the theory of acts, since it is not sufficiently comprehensive. However, its contents - particularly chapters one and two - are written in the spirit of a prolegomenon to that theory.

In these two chapters we give the definitions of admissible pair, due to Bednarek and Wallace [4],* and of left, right and two-sided ideals for acts. These last seem to be new. An admissible pair is the analogue for acts of the notion of congruence for semigroups, and the notion of ideal is fashioned upon the eponymous semigroup definition. In this spirit, we investigate quotients of acts by admissible pairs and in particular by closed ideals to obtain an analogue of the ideal theory for topological semigroups. The Noetherian isomorphism theorems are seen to hold as a consequence of an act-theoretic version of a theorem of Sierpinski on the completion of certain mapping diagrams, and a Schreier refinement theorem for discrete acts is given.

In the third chapter the category of acts is defined, and we establish some of the elementary properties of the category, such as the coreflectivity of the category of pairs of sets; from this follows the existence of coproducts. The most important result in this direction is that any

---

*Bracketed numbers are references to the bibliography.

compact totally disconnected act is the inverse limit of finite discrete acts. This extends the principal result of [5] from the subcategory of acts with a given input semigroup to the whole category of acts. Since the inverse limit of a system of sets is a subset of the cartesian product of the sets, this result may be considered as an embedding theorem; hence one may look for conditions under which each of the finite "approximants" - the term is due to Bednarek and Wallace [5] - is of some given cardinality. The embedding theorem gives conditions which are sufficient (and for the most part necessary) that a compact totally disconnected act be embedded in the product of two-point Boolean lattices acting on themselves via infimum multiplication.

The author would like to acknowledge the aid of his supervisory committee in the preparation of this dissertation; in particular he is indebted to Professor A. R. Bednarek, the committee chairman, for many conversations on matters mathematical. He is also indebted to Alice Anson for her impeccable typing. This research was supported by the department of mathematics of the University of Florida and by the National Science Foundation (GP6505).

1. PRELIMINARIES

1.1 Definitions

A semigroup is a Hausdorff space S with a continuous binary associative operation defined on it.

An act is a triple (S, X, m) with S a semigroup, X a topological space and $m: S \times X \to X$ such a continuous function that, letting $sx = m(s,x)$, $s(tx) = (st)x$. (We customarily denote the semigroup operation as well as the function m by juxtaposition unless the requirement of clarity dictates otherwise.) We consider m to be fixed once for all, and will refer to the act (S, X, m) simply as (S, X), supressing explicit mention of m wherever it is possible to do so without loss of clarity. In addition m will occasionally be referred to as the action of S on X; S is called the input and X the state space of (S, X). In the sequel, (S, X) will always denote an act if no hypothesis to the contrary is explicity stated.

Of course what is here called an act could have been called a left act, allowing room for the definition of the notion of a right act as a triple (X, S, m) where X is a Hausdorff space S is a semigroup and $m: X \times S \to S$ satisfies $m(m(x,s_1),s_2) = m(x,s_1s_2)$ for all $x \in X$ and all $s_1,s_2 \in S$. There is a rather obvious duality, of course, in the notions, which we formalize briefly. For any

semigroup $(S, \cdot)$, define the <u>dual</u> semigroup to be $(S, \circ)$ where $s \circ t = t \cdot s$. Let S' denote the dual of S, supressing mention of the operation on S. If now (S, X) is a (left) act, define its <u>dual</u> to be $(S, X)' = (X, S', *)$ where $x * s = sx$. It follows that $(S, X)'$ is a right act. If we make a similar definition for the dual of a right act (X, S) as $(X, S)' = (S', X, *)$ with an obvious definition of * then it follows that $(S, X, *)'' = (S, X, *)$ for any act (S, X). It can be seen that each theorem about (left) acts is logically equivalent to a "dual" theorem for right acts. We will rarely have use for this duality (Section 3.4 is an exception), so it will not be treated in any further detail.

By convention all sets are topologized with a Hausdorff topology. The closure, interior and boundary of a set A will be denoted by A^*, A° and $F(A)$ respectively; the empty set is denoted by $\emptyset$. For ease of expression we note that the collection $P(S) \times P(X)$ of pairs of subsets of S and of X is a lattice with $(A, B) \leq (C, D)$ iff $A \subseteq C \subseteq S$ and $B \subseteq D \subseteq X$. Hence the lattice operations are coordinatewise union and intersection for join and meet, respectively. Unless it is specifically stated otherwise, the symbols $\vee$ and $\wedge$ will always mean the lattice operations in $P(S) \times P(X)$, i.e. if $(A, B) \leq (S, X)$ and $(C, D) \leq (S, X)$ then

$$(A, B) \vee (C, D) = (A \cup C, B \cup D)$$

and

$$(A, B) \wedge (C, D) = (A \cap C, B \cap D).$$

A pair (A, B) in $P(S) \times P(X)$ will be called closed iff $A = A^*$ and $B = B^*$. Similar locutions employing other topological adjectives, such as compact or open for example, are defined analogously. If $(A, B) \leq (S, X)$ then $AB = \{ab \mid (a,b) \in A \times B\}$. It is an immediate consequence of the continuity of the action of S upon X that $A^*B^* \subseteq (AB)^*$ for all $(A, B) \leq (S, X)$, and $A^*B^* = (AB)^*$ if the action is also a closed function, which is the case if both S and X are compact.

If for each δ in a suitable index set Δ we have $(A_\delta, B_\delta) \leq (S, X)$ then it is clear that $(\bigcup_{\delta\in\Delta} A_\delta)(\bigcup_{\delta\in\Delta} B_\delta) = \bigcup_{(\delta,\delta')\in\Delta\times\Delta} A_\delta B_{\delta'}$ and $(\bigcap_{\delta\in\Delta} A_\delta)(\bigcap_{\delta\in\Delta} B_\delta) \subseteq \bigcap_{(\delta,\delta')\in\Delta\times\Delta} A_\delta B_{\delta'}$. We list a few examples next. Any semigroup S acts on itself via its multiplication, and any semigroup S acts on any space X via the identity $sx = x$. This action, which is just the projection of $S \times X$ onto X, satisfies the identities $s(tx) = t(sx)$ and $s(sx) = sx$. If, next, X is a locally compact space then as is well known, the set X^X of all continuous functions from X into itself is a semigroup in the compact-open topology which acts on X via evaluation: $m(f,x) = f(x)$ for each pair (f,x) in $X^X \times X$. Every topological transformation group [12]* is an act, as is every pair (S, I) where I is a left ideal of a semigroup S. In particular the action of S upon itself via its multiplication will be denoted by (S, S). If E is a left congruence on S,

---

*Bracketed numbers are references to the bibliography.

then S acts on S/E in an obvious way ([20], [21], [22], [3]).

A pair $(T, Y) \leq (S, X)$ is a <u>subact</u> if T is a subsemigroup of S and $TY \subseteq Y$. The state space of a subact (T, Y) is elsewhere called a <u>T-ideal</u> ([9], [8]). In particular, we consider $(\emptyset, \emptyset)$ to be a subact.

The <u>product</u> $\prod_{\alpha \in A} (S_\alpha, X_\alpha)$ of a family $\{(S_\alpha, X_\alpha) | \alpha \in A\}$ of acts is again an act with the action defined coordinatewise: $(sx)_\alpha = s_\alpha x_\alpha$, $\alpha \in A$. It is elementary that the action so defined is continuous. We will sometimes write $(S_1 \times S_2 \times \ldots \times S_n, X_1 \times X_2 \times \ldots \times X_n)$ to mean $\prod_{i=1}^{n} (S_i, X_i)$ if n = 2 or 3.

A <u>homomorphism</u> from (S, X, m) to an act (T, Y, m′) is such a pair (f,g) that f: S → T is a semigroup homomorphism (i.e. f is continuous and is algebraically a homomorphism) and g: X → Y is a continuous function and the diagram below commutes.

$$\begin{tikzcd}
S \times X \arrow[r, "m"] \arrow[d, "f \times g"] & X \arrow[d, "g"] \\
T \times Y \arrow[r, "m'"] & Y
\end{tikzcd}$$

(The function $f \times g$: $S \times X \to T \times Y$ is given by $f \times g(s,x) = (f(s), g(x))$ for each $(s,x) \in S \times X$.) If (f,g) is a homomorphism from (S, X) to (T, Y) we may write $(f,g): (S, X) \to (T, Y)$. We remark that if $\pi_\alpha: \prod_{\alpha \in A} S_\alpha \to S_\alpha$ and $\rho_\alpha: \prod_{\alpha \in A} X_\alpha \to X_\alpha$ are the canonical projection functions and (S_α, X_α) is an act for each $\alpha \in A$ then each pair $(\pi_\alpha, \rho_\alpha)$ is a homomorphism.

If we identify a function with its graph, the (f,g) notation for homomorphisms is justified.

Proposition 1.1. A homomorphism $(f,g): (S, X) \to (T, Y)$ is a closed subact of $(S \times T, X \times Y)$.

Proof. If $((s,f(s)),(x,g(x)))$ is any element of $f \times g$ then $(sx,f(s)g(x)) = (sx,g(sx))$ is in the graph of g, i.e. $fg \subseteq g$. The continuity of f and g and the fact that T and Y are Hausdorff imply that the graphs of f and g are closed, as is well-known. It is convenient to recall that a congruence C on a semigroup S is an equivalence relation on S which is a subsemigroup of $S \times S$. We will call a pair (F, E) admissible on (S, X) if F is a congruence on S, E is an equivalence on X and (F, E) is a subact of $(S \times S, X \times X)$. Proposition 1.3 below appears in [4], but is repeated here for the sake of completeness. It is a corollary to a theorem of Sierpinski, part of which we state here without proof.

Proposition 1.2. (Sierpinski). If, in the following diagram of topological spaces and continuous functions,

$$\begin{tikzcd} X \arrow[rr, "\phi"] \arrow[dr, "\psi"'] & & Y \\ & Z \arrow[ur, dashed, "h"'] & \end{tikzcd}$$

the conditions

(i) $Z = \psi(X)$

(ii) $\psi(x) = \psi(y)$ implies $\phi(x) = \phi(y)$

(iii) X compact

are satisfied, then there is a unique continuous function h making the diagram commute. Furthermore if $Y = \phi(X)$ and the converse of (ii) holds then h is a homeomorphism.

Clearly if ϕ and ψ are homomorphisms of semigroups then h is also one such. We shall use 1.2 to prove an analogous result for acts.

<u>Proposition 1.2´</u>. If, in the following diagram of acts and homomorphisms,

$$\begin{tikzcd}
(S, X) \arrow[rr, "{(f_2,g_2)}"] \arrow[dr, "{(f_1,g_1)}"'] & & (T, Y) \\
& (U, Z) \arrow[ur, dashed, "{(h_1,h_2)}"'] &
\end{tikzcd}$$

the conditions

(i)´ $f_1(S) = U$ and $g_1(X) = Z$

(ii)´ $f_1(s) = f_1(t)$ implies $f_2(s) = f_2(t)$ and $g_1(x) = g_1(y)$ implies $g_2(x) = g_2(y)$

(iii)´ S and X are compact

then there is a unique homomorphism (h_1,h_2) making the diagram commutative.

<u>Proof</u>. The existence of h_1 and h_2 follow from two applications of 1.2; to see that (h_1,h_2) is a homomorphism, let $(u,z) \in U \times Z$, $(u,z) = (f_1(s),g_1(x))$ say. Then $h_1(u)h_2(z) = (h_1f_1(s))(h_2g_1(x)) = f_2(s)g_2(x) = g_2(sx) = h_2g_1(sx) = h_2(f_1(s)g_1(x)) = h_2(uz)$.

<u>Proposition 1.3</u>. If (F, E) is a closed admissible pair on a compact or discrete act (S, X) then there is a unique action of S/F upon X/E for

which the pair (f,g) of natural surjections of S upon S/F and X upon X/E is a homomorphism.

<u>Proof</u>. Since F and E are closed then, as is well-known, the compactness (respectively discreteness) of S and of X imply that the quotient spaces S/F and X/E are compact and Hausdorff (respectively discrete). It is easy to see that in the following diagram (i) and (ii) of 1.2 hold with $\psi = f \times g$ and $\phi = gm$. Hence there is a unique continuous m' making the diagram commute.

$$\begin{tikzcd} S \times X \arrow[r, "m"] \arrow[d, "f \times g"] & X \arrow[d, "g"] \\ S/F \times X/E \arrow[r, "m'"] & X/E \end{tikzcd}$$

Writing $m'(\overline{s},\overline{x}) = \overline{s} * \overline{x}$ we see that since m is an action,

$f(s) * (f(t) * g(x)) = f(s) * g(tx) = g(s(tx)) = g((st)x) = f(st) * g(x)$, i.e. $f(s) * (f(t) * g(x)) = f(st) * g(x)$ (1)

Now in order to conclude that m' is an action, we need to show that S/F is a semigroup with such an operation $\circ$ that $f(st) = f(s) \circ f(t)$. But surely we may apply our result (1) in the case $X = S$ and $E = F$. Hence from Sierpinski's theorem we obtain a function $\circ$ such that $f(s) \circ (f(t) \circ f(x)) = f(st) \circ f(x)$ and the diagram below commutes

$$\begin{tikzcd} S \times S \arrow[r] \arrow[d, "f \times f"] & S \arrow[d, "f"] \\ S/F \times S/F \arrow[r, "\circ"] & S/F \end{tikzcd}$$

i.e. $f(st) = f(s) \circ f(t)$. (2)

Hence S/F is a semigroup and from (1) and (2) we may conclude

$$f(s) * (f(t) * g(x)) = (f(s) \circ f(t)) * g(x),$$

i.e. $*$ is an action.

The act so defined is called the <u>quotient</u> of (S, X) modulo the admissible pair (F, E).

1.2 Some Isomorphism Theorems

The next lemma is a special case of a categorical theorem, the general formulation of which would take us too far afield at present. Another special case, that of universal algebra, appears as Lemma II.3.1 in Cohn's Universal Algebra [7]. We pause to give some notation. If A, B and C are sets and R is a relation from A to B, i.e. $R \subseteq A \times B$, then $R^{-1} = \{(y,x) \mid (x,y) \in R\} \subseteq B \times A$ and if also $S \subseteq B \times C$ then $S \circ R = \{(x,y) \mid \exists z \cdot (x,z) \in R$ and $(z,y) \in S\}$. For any family of sets $\{Z_i \mid i \in I\}$ the jth projection from $\prod_{i \in I} Z_i$ onto Z_j is denoted by π_j. If $A' \subseteq A$ and $B' \subseteq B$ then $A'R = \pi_2((A' \times B) \cap R) = \{y \mid \exists\, x \in A' \cdot (x,y) \in R\}$; dually $RB' = \pi_1((A \times B') \cap R) = \{x \mid \exists\, y \in B' \cdot\ni\cdot (x,y) \in R\}$, and $R|A' = R \cap (A' \times A')$.

Lemma 1.4. Let (S_i, X_i), $i \in \{1, 2, 3\}$ be acts with (T, Y) a subact of (S_1, X_1) and (Φ_i, Ψ_i) a subact of $(S_i \times S_{i+1}, X_i \times X_{i+1})$, for $i = 1, 2$. Then $(T\Phi_1, Y\Psi_1)$, $(\Phi_i^{-1}, \Psi_i^{-1})$ and $(\Phi_2 \circ \Phi_1, \Psi_2 \circ \Psi_1)$ are subacts of (S_2, X_2), $(S_{i+1} \times S_i, X_{i+1} \times X_i)$ and $(S_1 \times S_3, X_1 \times X_3)$, respectively.

Proof. Since π_2 is a semigroup homomorphism, then $T\Phi_1 = \pi_2((T \times S_2) \cap \Phi_1)$ is a semigroup; now if $s \in T\Phi_1$ and $x \in Y\Psi_1$ then for some $s' \in T$ and for some $x' \in Y$, $(s',s) \in \Phi_1$ and $(x',x) \in \Psi_1$. From the hypothesis, $sx \in TY \subseteq Y$ and $(s',s)(x',x) = (s'x',sx) \in \Phi_1\Psi_1 \subseteq \Psi_1$ so that $sx \in Y\Psi_1$,

proving the first assertion. If $(sx,ty) = (s,t)(x,y) \in \Phi_1^{-1}\Psi_1^{-1}$ then $(t,s) \in \Phi_1$ and $(y,x) \in \Psi_1$, so that $(ty,sx) \in \Phi_1 \Psi_1 \subseteq \Psi_1$ whence it follows that $(sx,ty) \in \Psi_1^{-1}$, proving the second assertion. If now $(sx,ty) \in (\Phi_1 \circ \Phi_2, \Psi_1 \circ \Psi_2)$ then for some u and for some z $(s,u) \in \Phi_1$, $(u,t) \in \Phi_2$, $(x,z) \in \Psi_1$ and $(z,y) \in \Psi_2$. Hence $(sx,uz) \in \Phi_1 \Psi_1 \subseteq \Psi_1$ and similarly $(uz,ty) \in \Psi_2$, so that $(\Phi_2 \circ \Phi_1)(\Psi_2 \circ \Psi_1) \subseteq \Psi_2 \circ \Psi_1$ which proves the third assertion.

Corollary 1.5. (i) The composition of homomorphisms is a homomorphism; (ii) The homomorphic image of an act is a subact.

Proof. In view of 1.1, in the preceding lemma take (Φ_1, Ψ_1) to be a homomorphism.

If $f\colon X \to Y$ is a function, its kernel $K(f) = \{(x,y) \mid f(x) = f(y)\}$. The kernel of any function is an equivalence relation which, furthermore, is closed in $X \times X$ provided that Y is a Hausdorff space and f is continuous; $K(f)$ is a congruence if f is a semigroup homomorphism.

Proposition 1.6. If (f,g) is a homomorphism of (S, X) to an act (T, Y), then $(K(f), K(g))$ is an admissible pair.

Proof. If $f(s) = f(s')$ and $g(x) = g(x')$ then $f(sx) = f(s)g(x) = f(s')g(x') = f(s'x')$, so that $K(f)K(g) \subseteq K(g)$; in view of the immediately preceding remarks the assertion then follows. We will sometimes use the slightly simpler notation $K(f,g)$ to denote $(K(f), K(g))$. A homomorphism (f,g) is an isomorphism if f and g are bijections. We may now formulate analogues of the classical isomorphism theorems.

Proposition 1.7. (First Isomorphism Theorem) If $(f,g)\colon (S, X) \to (T, Y)$ is a homomorphism of compact or discrete acts then $(f,g) = (i\overline{f}\rho, j\overline{g}\sigma)$, where $i\colon f(S) \to T$ and $j\colon g(X) \to Y$ are inclusion maps, $(\rho,\sigma)\colon (S, X) \to (S/K(f), X/K(g))$ is the natural projection and $(\overline{f},\overline{g})\colon (S/K(f), X/K(g)) \to (f(S), g(X))$ is an isomorphism.

$$\begin{tikzcd}
(S, X) \arrow[rr, "{(f,g)}"] \arrow[d, "{(\rho,\sigma)}"] & & (T, Y) \\
(S/K(f), X/K(g)) \arrow[rr, "{(\overline{f},\overline{g})}"] & & (f(S), g(X)) \arrow[u, "{(i,j)}"']
\end{tikzcd}$$

Proof. 1.6 implies that $(K(f), K(g))$ is an admissible pair and then from 1.3 it follows that $(S/K(f), X/K(g))$ is an act. 1.5 (ii) implies that $(f(S), g(X))$ is an act, in fact a subact of (T, Y), and surely the pair (i,j) of inclusions is a homomorphism. The existence of $(\overline{f},\overline{g})$ with the asserted properties follows from 1.2′.

We remark that the factorization of (f,g) into a composition of injective, bijective, and surjective homomorphisms is unique up to isomorphism. The proof of this remark is straightforward but tedious and is omitted, as is the proof of 1.8 below, which is an analogue of the classical second isomorphism theorem.

Proposition 1.8. (Second Isomorphism Theorem) If (T, Y) is a closed subact of a compact or discrete act (S, X) and (F, E) is a closed admissible pair on (S, X), then (in the notation of 1.4) (TF, YE) is a closed subact of (S, X) and

$$(TF/F|TF,\ YE/E|YE) \simeq (T/F|T,\ Y/E|Y).$$

The third isomorphism theorem is an immediate consequence of 1.2´. Recall that $(A, B) \leq (C, D)$ iff $A \subseteq C$ and $B \subseteq D$.

<u>Proposition 1.9</u>. (Third Isomorphism Theorem)

If (F_i, E_i), $i = 1, 2$ are closed admissible pairs on a compact or discrete act (S, X), and if $(F_1, E_1) \leq (F_2, E_2)$, then there is a unique homomorphism (f,g) making a commutative diagram with the natural projections, as in the following diagram.

$$\begin{tikzcd}
(S, X) \arrow[rr] \arrow[dr] & & (S/F_2, X/E_2) \\
& (S/F_1, X/E_1) \arrow[ur, dashed, "(f,g)"] &
\end{tikzcd}$$

1.3 A Schreier Refinement Theorem

A pair $(T, Y) \leq (S, X)$ is called a left (respectively right) ideal of (S, X) if T is a left (respectively right) ideal of S and $SY \subseteq Y$ (respectively $TX \subseteq Y$). We say that (T, Y) is an ideal of (S, X) if it is both a left ideal and a right ideal. We notice that any (left or right) ideal is a subact.

Associated with an ideal (T, Y) of (S, X) is an admissible pair (F, E) called the Rees pair for (T, Y) given by $F = T \times T \cup \Delta_S$ and $E = Y \times Y \cup \Delta_X$ (for any set A, $\Delta_A = \{(x,x) \mid x \in A\}$). Verification that (F, E) is admissible is routine. If now (S, X) is compact or discrete and (T, Y) is a closed ideal then the Rees pair for (T, Y) is closed. The quotient of (S, X) by this admissible pair is called the Rees quotient and is usually written $(S/T, X/Y)$ rather than $(S/F, X/E)$. A series from an act (S, X) to a subact (T, Y) is such a sequence (S_i, X_i), $0 \leq i \leq n$ of subacts of (S, X) that $(S_0, X_0) = (S, X)$, $(S_n, X_n) = (T, Y)$ and (S_i, X_i) is an ideal of (S_{i-1}, X_{i-1}) for each $i = 1, 2, \ldots, n$. The Rees quotients $(S_{i-1}/S_i, X_{i-1}/X_i)$ are called the factors of the series. A refinement of a series $(S_i, X_i)_{i=0}^{n}$ from (S, X) to (T, Y) is such a series $(T_j, Y_j)_{j=0}^{p}$ whose terms include all the terms of $(S_i, X_i)_{i=1}^{n}$. Two series are called isomorphic if there is a 1-1 correspondence between them with paired factors isomorphic as acts.

Theorem 1.10. Any two series from (S, X), a discrete act, to a subact (T, Y) have isomorphic refinements.

Proof. If $\{(S_i, X_i) | 0 \leq i \leq n\}$ and $\{(T_j, Y_j) | 0 \leq j \leq p\}$ are two series from (S, X) to (T, Y), we define $(S_{ij}, X_{ij}) = (S_i \cup (S_{i-1} \cap T_j), X_i \cup (X_{i-1} \cap Y_j))$ for all (i,j) such that $1 \leq i \leq n$ and $0 \leq j \leq p$; dually we define $(T_{ji}, Y_{ji}) = (T_j \cup (T_{j-1} \cap S_i), Y_j \cup (Y_{j-1} \cap X_i))$ for all (j,i) such that $1 \leq j \leq p$ and $1 \leq i \leq n$. Then $(S, X) = (S_{10}, X_{10}) \supseteq (S_{11}, X_{11}) \supseteq \ldots \supseteq (S_{1p}, X_{1p}) = (S_1, X_1) = (S_{20}, X_{20}) \supseteq \ldots \supseteq (S_{np}, X_{np}) = (T, Y)$ and $(S, X) = (T_{10}, Y_{10}) \supseteq (T_{11}, Y_{11}) \supseteq \ldots \supseteq (T_{1n}, Y_{1n}) = (T_1, Y_1) = (T_{20}, Y_{20}) \supseteq (T_{21}, Y_{21}) \supseteq \ldots \supseteq (T_{pn}, Y_{pn}) = (T, Y)$. It is well known [6], [18] that $(S_{i,j})$ is an ideal of $S_{i,j-1}$ for each i,j for each i,j such that $1 \leq j \leq p$ with an obvious dual remark applying to each $T_{j,i}$. To see that (S_{ij}, X_{ij}) is an ideal of $(S_{i,j-1}, X_{i,j-1})$ it is enough to show $S_{i,j-1} \ X_{ij} \subseteq X_{ij}$ and $S_{ij} \ X_{i,j-1} \subseteq X_{ij}$. The verifications are routine. Since (S_i, X_i) is an ideal of (S_{i-1}, X_{i-1}) for each i and (T_j, Y_j) is an ideal of (T_{j-1}, Y_{j-1}) we have $S_{i-1} X_i \cup S_i X_{i-1} \subseteq X_i$ and $T_{j-1} Y_j \cup T_j Y_{j-1} \subseteq Y_j$. Then $S_{i,j-1} X_{ij} = (S_i \cup (S_{i-1} \cap T_{j-1}))(X_i \cup (X_{i-1} \cap Y_j)) = S_i X_i \cup S_i(X_{i-1} \cap Y_j) \cup (S_{i-1} \cap T_{j-1}) X_i \cup (S_{i-1} \cap T_{j-1})(X_{i-1} \cap Y_j) \subseteq X_i \cup (S_{i-1} X_{i-1} \cap T_{j-1} Y_j) \subseteq X_i \cup (X_{i-1} \cap Y_j) = X_{ij}$, so (S_{ij}, X_{ij}) is a left ideal of $(S_{i,j-1}, X_{i,j-1})$. That it is also a right ideal, and hence an ideal, is verified in a similar fashion. Now the quotient semigroups $S_{i,j-1}/S_{ij}$ and $T_{j,i-1}/T_j$, as is well-known (see [6], e.g.), are not only isomorphic but identical if their respective zero elements be identified, so that an explicit semigroup isomorphism f_{ij} is at hand. Indeed, if 0 (respectively $0'$) is the zero of $S_{i,j-1}/S_{ij}$ (respectively $T_{j,i-1}/T_{ji}$) then

$S_{i,j-1}/S_{ij} = O \cup S_{i,j-1}\backslash S_{ij}$ and $T_{j,i-1}/T_{ji} = O' \cup T_{j,i-1}\backslash T_{j,i}$. But $S_{i,j-1}\backslash S_{ij} = S_{i-1}\backslash S_i \cap T_{j-1}\backslash S_i \cap S_{i-1}\backslash T_j \cap T_{j-1}\backslash T_j = T_{j,i-1}\backslash T_{j,i}$. Since this observation is set-theoretic rather than algebraic, it follows mutatis mutandis that there is similarly an explicit one-to-one correspondence g_{ij} between $X_{i,j-1}/X_{ij}$ and $Y_{j,i-1}/Y_{ji}$. It is routine and elementary that (f_{ij}, g_{ij}) is an isomorphism.

It is not known if 1.10 holds with "discrete" replaced by "compact" and "subact" replaced by "closed subact."

1.4 Some Lattices Associated with an Act

Let $I(S, X)$ denote the collection of all nonvoid ideals of (S, X). Then $I(S, X)$ is itself nonvoid since it contains (S, X). Now if (S_i, X_i), $i = 1, 2$, are ideals we have $\emptyset \neq S_1S_2 \subseteq S_1 \cap S_2$, and $\emptyset \neq S_2X_1 \subseteq SX_1 \cap S_2X \subseteq X_1 \cap X_2$; furthermore, $S(X_1 \cap X_2) \cup (S_1 \cap S_2) X \subseteq X_1 \cap X_2$, hence $(S_1 \cap S_2, X_1 \cap X_2)$ is an ideal of (S, X), and is the greatest lower bound in $P(S) \times P(X)$ of (S_1, X_1) and (S_2, X_2). Similarly we see that $(S_1 \cup S_2, X_1 \cup X_2)$ is an ideal of (S, X) and is the least upper bound in $P(S) \times P(X)$ of (S_1, X_1) and (S_2, X_2). It follows that $I(S, X)$ is a distributive sublattice of $P(S) \times P(X)$.

Next let $A(S, X)$ denote the collection of all subacts of (S, X). As before, $(S, X) \in A(S, X)$ so this collection is nonvoid, and since the intersection of any nonvoid collection of subacts is a subact (we do not require that the statespace or input of an act be nonvoid), we have that $A(S, X)$ is a complete infimum semilattice of $P(S) \times P(X)$. Since (S, X) is the least upper bound of $A(S, X)$ and is a member of $A(S, X)$, then $A(S, X)$ is a complete lattice. The supremum operation may be defined as $(S_1, X_1) \vee (S_2, X_2) = \wedge \{(T, Y) \in A(S, X) \mid (S_1 \cup S_2, X_1 \cup X_2) \leq (T, Y)\}$. It is clear that we may define the subact $[(A, B)]$ generated by any pair $(A, B) \leq (S, X)$ as above, $[A, B] = \wedge \{(T, Y) \in A(S, X) \mid (A, B) \leq (T, Y)\}$. On the other hand, for $A \subseteq S$ let $[A]$ be the

subsemigroup of S generated by A, and for $B \subseteq X$ let $[B] = \cap\{Z \subseteq X \mid B \subseteq Z$ and $[A]\ Z \subseteq Z\}$. Then it is easy to see that $[(A, B)] = ([A], [B])$. Since $[(S_1 \cup S_2, X_1 \cup X_2)] = (S_1 \cup S_2, X_1 \cup X_2)$ for all ideals (S_1, X_1) and (S_2, X_2) it is clear that $I(S, X)$ is a sublattice of $A(S, X)$.

If we denote by $C(S)$ the lattice of all closed congruences on a compact or discrete semigroup S and by $E(X)$ the lattice of all closed equivalence relations on a compact or discrete space X, then the subset $C(S, X)$ of $C(S) \times E(X)$ consisting of all admissible pairs on (S, X) is of some interest. To see that $C(S, X)$ is a sublattice of $C(S) \times E(X)$, we need only remark that (Δ_S, Δ_X) is a closed admissible pair which is contained in every admissible pair and hence that $C(S, X)$ is a complete infimum semilattice; since $(S \times S, X \times X)$ is a maximal element of $C(S, X)$, it follows that $C(S, X)$ is a complete lattice. Moreover, in the discrete case we can describe $(F_1, E_1) \vee (F_2, E_2)$ for any pair (F_i, E_i), $i = 1, 2$ of elements in $C(S, X)$ with the aid of the following lemma.

<u>Lemma 1.11</u>. If (S, X) is a discrete act and $(F_i, E_i) \in C(S, X)$ for $i = 1, 2$ then $\Delta_S(E_1 \vee E_2) \subseteq E_1 \vee E_2$ and $(F_1 \vee F_2)\ \Delta_X \subseteq E_1 \vee E_2$.

<u>Proof</u>. It is well-known that $(x,y) \in E_1 \vee E_2$ if and only if there is a finite sequence $x = x_0, x_1, \ldots, x_n = y$ such that $(x_{j-1}, x_j) \in E_1 \cup E_2$ for each j, $1 \leq j \leq n$. Now if $s \in S$ and $(x,y) \in E_1 \vee E_2$ then $\Delta_S\ E_i \subseteq F_i\ E_i \subseteq E_i$ for $i = 1, 2$, hence for each j, $(sx_{j-1}, sx_j) \in E_1 \cup E_2$, i.e. $(sx,sy) \in E_1 \vee E_2$, proving the first assertion. The

second assertion is proved similarly. Now it is clear that $(F_1 \vee F_2, E_1 \vee E_2) = (F_1, E_1) \vee (F_2, E_2)$ for if $(s,t) \in F_1 \vee F_2$ and $(x,y) \in E_1 \vee E_2$ then (sx,sy) and (sy,ty) are in $E_1 \vee E_2$ by 1.11, and since $E_1 \vee E_2$ is transitive, $(sx,ty) \in E_1 \vee E_2$, so $(F_1 \vee F_2, E_1 \vee E_2)$ is an admissible pair; the result now follows.

2. RELATIVE IDEALS IN ACTS

We begin by generalizing the notion of ideal. If (T, Y) is a subact of (S, X), then a pair $(A, B) \leq (S, X)$ is called a <u>left</u> (respectively <u>right</u>) (T, Y)-ideal if the following conditions hold:

$TB \subseteq B$ (respectively $AY \subseteq B$)

and

A is a left (right) T-ideal.

(A is a left (right) T-ideal if $TA \subseteq A$, ($AT \subseteq A$) [21], [22], [3].) (A, B) is a $(T\text{-}Y)$ <u>ideal</u> iff it is both a left and a right (T, Y) ideal. A (T, Y)-ideal will be referred to generically as a <u>relative act ideal</u> or just relative ideal if no confusion with the semigroup analogue is possible. In this chapter we will state and prove some results concerning the existence of minimal and maximal proper relative ideals. We remark that an ideal of (S, X) as defined in section 1.3 is an (S, X)-ideal in the present definition. It is furthermore easy to see that the collection of all nonvoid (T, Y)-ideals for fixed $(T, Y) \leq (S, X)$ is a sublattice of $P(S) \times P(X)$. The proof parallels that of the analogous remark in section 1.4. We will use the following notation adopted from the theory of semigroups ([21], [22], [3]).

If $A \subseteq S$ and T is a subsemigroup of S, let

$$L(A) = A \cup TA$$

$$R(A) = A \cup AT$$

$$J(A) = A \cup AT \cup TA \cup TAT.$$

Juxtaposition here denotes multiplication in S.

If $B \subseteq X$, define

$$L(B) = B \cup TB,$$

juxtaposition now denoting the action of S upon X. It will be clear from context in which sense L is being used.

2.1 Minimal Relative Ideals

For any pair $(A, B) \leq (S, X)$ define the <u>left</u> (T, Y)-ideal <u>generated</u> by (A, B) to be

$$L(A, B) = \wedge \{(P, Q) \mid (A, B) \leq (P, Q) \text{ and } (P, Q) \text{ is a left } (T, Y)\text{-ideal}\}.$$

The right (T, Y) ideal $R(A, B)$ and the <u>ideal</u> $J(A, B)$ generated by (A, B) are defined analogously.

A (T, Y)-ideal (respectively left ideal, right ideal) is called <u>minimal</u> iff it is minimal, with respect to the partial order on $P(S) \times P(X)$, among all (T, Y)-ideals (respectively all left (T, Y)-ideals, all right (T, Y)-ideals). For the following lemma let $(A, B) \leq (S, X)$ and let (T, Y) be a fixed subact of (S, X).

<u>Lemma 2.1</u>. (i) $J(A, B) = (J(A), L(B) \cup J(A)Y)$

(ii) $L(A, B) = (L(A), L(B))$

(iii) $R(A, B) = (R(A), B \cup R(A)Y)$

<u>Proof</u>. We prove only (i); the remaining assertions are proved similarly. It is clear that $J(A, B)$ is a (T, Y)-ideal containing (A, B); if on the other hand (P, Q) is any (T, Y)-ideal containing (A, B) then $J(A) \subseteq P$ since P is an ideal of S, and furthermore $B \cup TB \cup J(A)Y \subseteq Q \cup TQ \cup PY \subseteq Q$ so that $J(A, B) \leq (P, Q)$. The assertion (i) is hence proved. The next result is the act analogue of a basic result in semigroups [13].

Theorem 2.2. If (T, Y) is a closed subact of a compact act (S, X) then there is a unique minimal (T, Y)-ideal; the state space and input of this minimal ideal are closed.

Proof. Let $(C, \leq)$ be the collection of all closed (T, Y)-ideals (i.e. both the input and the state space are closed sets), with the partial order inherited from $P(S) \times P(X)$. (T, Y) is such a closed (T, Y)-ideal, hence C is nonvoid. If C' is any chain in C and if $(P, Q) = \wedge C'$ (recall that infimum in $P(S) \times P(X)$ is coordinatewise set intersection), then P and Q are each the intersection of a chain of closed sets, so by the compactness of S and of X, $P^* = P \neq \emptyset \neq Q = Q^*$. If now $(F, G) \in C'$ then $(P, Q) \leq (F, G)$, hence $TQ \cup PY \subseteq TG \cup FY \subseteq G$; since (F, G) is arbitrary, it follows that $TQ \cup PY \subseteq Q$, i.e. (P, Q) is a closed (T, Y)-ideal, and is the lower bound of the chain C'. Since we have proved that every nonvoid chain in C has a lower bound, we may conclude from Zorn's lemma that C has a minimal element, (J, K) say. If now (A, B) is any (T, Y)-ideal, not necessarily closed, and if $(A, B) \leq (J, K)$ then for any point $(s,x) \in A \times B$ we have first that $J(s) = \{s\} \cup sT \cup Ts \cup TsT \subseteq A$ since A is a T-ideal and next that $L(x) \cup J(s)\,Y = x \cup Tx \cup J(s)\,Y \subseteq B \cup TB \cup AY \subseteq B$; but $(J(s), L(x) \cup J(s)\,Y)$ is the ideal generated by (s,x), and is closed; hence $(J, K) \leq (J(s), L(x) \cup J(s)\,Y) \leq (A, B) \leq (J, K)$. It follows that $(J, K) = (A, B)$, i.e. (J, K) is minimal among all (T, Y)-ideals. We observe next that $(J, K) \leq (T, Y)$ since evidently $(T \cap J, Y \cap K)$ is a (T, Y)-ideal and therefore

contains (J, K). In particular, J is a semigroup for $J^2 \subseteq TJ \subseteq J$. To see that (J, K) is unique as asserted let (J', K') be any minimal (T, Y)-ideal; then $\emptyset \neq JJ' \subseteq TJ' \cap JT \subseteq J' \cap J$, so $J' \cap J$ is a T-ideal. Furthermore $\emptyset \neq (J \cap J')Y \subseteq JY \cap J'Y \subseteq K \cap K'$ and hence $T(K \cap K') \cup (J \cap J')Y \subseteq (TK \cap TK') \cup K \cap K' \subseteq K \cap K'$ so $(J \cap J', K \cap K')$ is a (T, Y)-ideal and is contained in both (J, K) and (J', K'). The minimality of the latter then imply $J = J \cap J' = J'$ and $K = K \cap K' = K'$ so $(J', K') = (J, K)$, proving uniqueness. We remark that the uniqueness does not depend on any topological hypothesis on (S, X). It is easy to see that in any act (S, X) if (P, Q) is the minimal (T, Y)-ideal for any (T, Y), then P is necessarily the minimal T-ideal ideal of (S, X). Indeed if P' is a T-ideal contained in P, then $TQ \cup P'Y \subseteq TQ \cup PY \subseteq Q$, so that (P', Q) is a (T, Y)-ideal, and hence $P = P'$. Conversely if S has a minimal T-ideal J then (J, JY) is surely a (T, Y)-ideal, for $T(JY) \cup JY \subseteq (TJ)Y \cup JY \subseteq JY$. If now (P, Q) is a (T, Y)-ideal and $(P, Q) \leq (J, JY)$ then $P = J$ by minimality and $Q \subseteq JY$. On the other hand $Q \supseteq TQ \cup PY$ so it follows $Q = JY$, proving that (J, JY) is a minimal (T, Y)-ideal. In summary, we have proved the following proposition.

<u>Proposition 2.3</u>. An act (S, X) has a minimal (T, Y)-ideal iff S has a minimal T-ideal; if J is the minimal T-ideal, then (J, JY) is the minimal (T, Y)-ideal.

We remark that if S is a semigroup with a subsemigroup T and a minimal T-ideal J then the minimal (T, T)-ideal of (S, S) is (J, J). For by hypothesis $JT \subseteq J$, and from 2.3

it follows that (J, JT) is the minimal (T, T)-ideal, so we need only show $J \subseteq JT$. But this is trivial since JT is a T-ideal and hence contains the minimal T-ideal J.

If now (S, X) is any act and $(s, x) \in S \times X$, define $\Gamma_1(s) = \{s^n | n \geq 1\}^*$ and $\Gamma_2(s,x) = \{x\} \cup \{s^n x | n \geq 1\}^*$. From the continuity of the action it follows that $\Gamma_1(s)\Gamma_2(s,x) \subseteq \Gamma_2(s,x)$ with equality holding if the action of S upon X is a closed function, e.g. (S, X) is a compact act. Hence $(\Gamma_1(s), \Gamma_2(s,x))$ is a subact of (S, X). From theorem 2.2, if $(\Gamma_1(s), \Gamma_2(s,x))$ is compact, it has a minimal ideal (K, J) and since $\Gamma_1(s)$ is an abelian semigroup K must be a group; (K, J) is hence a compact group acting on a compact space.

2.2 Maximal Proper Relative Ideals

Maximal proper ideals in semigroups were studied by Koch and Wallace [14].

If $(A, B) \leq (S, X)$ and $C \subseteq X$ we define

$$A^{[-1]}B = \{x \in X \mid Ax \subseteq B\}$$

$$A^{(-1)}B = \{x \in X \mid Ax \cap B \neq \emptyset\}$$

$$BC^{[-1]} = \{s \in S \mid sC \subseteq B\}$$

$$BC^{(-1)} = \{s \in S \mid sC \cap B \neq \emptyset\}$$

The notation will frequently occur in the sequel. The following facts are well-known and proofs are available in many places, e.g. [2], [5].

Lemma 2.4. Let $(A, B) \leq (S, X)$

(i) If A is compact and B is closed then $A^{(-1)}B$ is closed

(ii) If A is compact and B is open then $A^{[-1]}B$ is open

(iii) If B is open then $A^{(-1)}B$ is open

(iv) If B is closed then $A^{[-1]}B$ is closed.

Similar assertions of course hold for $BC^{[-1]}$ and $BC^{(-1)}$ for $B, C \subseteq X$.

Lemma 2.5. If (T, Y) is a subact of (S, X) and (T, J) is a subact of (S, X) then $JY^{[-1]}$ is a T-ideal.

Proof. If $s \in JY^{[-1]}$ then $sY \subseteq J$. If also $t \in T$ then $(ts)Y = t(sY) \subseteq tJ \subseteq TJ \subseteq J$ so $ts \in JY^{[-1]}$. Also $(st)Y = s(tY) \subseteq s(TY) \subseteq sY \subseteq J$ so $st \in JY^{[-1]}$, proving the assertion.

For $(A, B) \leq (S, X)$ the largest (T, Y)-ideal contained in (A, B) is $(M(A), M(A, B)) = \vee \{(P, Q) | (P, Q)$ is a (T, Y)-ideal and $(P, Q) \leq (A, B)\}$. Since $M(A)$ in particular is a T-ideal it is contained in $\overline{M}(A)$, the maximal T-ideal contained in A. We give next a condition for which $M(A) = \overline{M}(A)$.

<u>Proposition 2.6</u>. In the above notation, a necessary and sufficient condition that $M(A) = \overline{M}(A)$ is that $\overline{M}(A) \subseteq M(A, B)Y^{[-1]}$.

<u>Proof</u>. ($\Longrightarrow$) It is no loss to suppose (A, B) contains at least one (T, Y)-ideal for if not it follows that $\overline{M}(A) = \emptyset$ and the proposition holds. So we suppose $M(A) \neq \emptyset \neq M(A, B)$, and since $(M(A), M(A, B))$ is a (T, Y)-ideal, then $T \cdot M(A) \cup M(A) \cdot Y \subseteq M(A, B)$ so that $M(A) \subseteq M(A, B)Y^{[-1]}$. Hence the condition is necessary. To prove sufficiency, we use the hypothesis to conclude $T \cdot M(A, B) \cup \overline{M}(A)Y \subseteq M(A, B)$ and hence we deduce that $(\overline{M}(A), M(A, B))$ is a (T, Y)-ideal, which implies $(\overline{M}, A), M(A, B) \leq (M(A), M(A, B))$. In particular $\overline{M}(A) \subseteq M(A)$ and we are finished.

<u>Lemma 2.7</u>. (i) If $(A, B) = (A^*, B^*)$ then $M(A)$ and $M(B)$ are closed.

(ii) If (S, X) is compact, (T, Y) is a closed subact of (S, X) and $(A, B) = (A^\circ, B^\circ)$ then $M(A)$ and $M(A, B)$ are open sets.

<u>Proof</u>. $(M(A)^*, M(A, B)^*) \leq (A^*, B^*) = (A, B)$ implies $M(A)^* \subseteq M(A)$ and $M(A, B)^* \subseteq M(A, B)$, proving the first assertion. To prove the second assertion, suppose $(t,x) \in M(A) \times M(A,B)$. We may assume without loss of generality

that there is a (T, Y)-ideal (P, Q) such that $(t,x) \in P \times Q$. Indeed from the definition of $(M(A), M(A, B)$ there are (T, Y)-ideals (P_1, Q_1) and (P_2, Q_2) contained in (A, B) such that $t \in P_1$ and $x \in Q_2$. But $(P_1 \cup P_2, Q_1 \cup Q_2)$ is also a (T, Y)-ideal contained in (A, B), and $(t,x) \in (P_1 \cup P_2) \times (Q_1 \cup Q_2)$. Since (P, Q) is a (T, Y)-ideal then $J(t,x) \leq (P, Q)$, i.e. $J_1(t) = t \cup Tt \cup tT \cup TtT \subseteq P \subseteq A = A°$ and $J_2(y) = y \cup Ty \cup J_1(t) \cdot Y \subseteq Q \subseteq B = B°$. By repeated application of Wallace's theorem [13] to the various compact sets whose union is $J_1(t)$ and $J_2(y)$ it follows that there are open sets U about t and V about x such that $J(U, V) \leq (A, B)$ so that $J(U, V) \leq (M(A), M(A, B))$. Since $(U, V) \leq J(U, V)$ it follows that $t \in U \subseteq M(A)$ and $x \in V \subseteq M(A, B)$; since t and x were chosen arbitrarily, we have shown that $M(A)$ and $M(A, B)$ are open.

We say that a pair (A, B) is <u>properly</u> contained in (S, X), $(A, B) < (S, X)$ if $S \setminus A \times X \setminus B \neq \emptyset$.

<u>Theorem 2.8</u>. If (T, Y) is a closed subact of a compact act (S, X) then each proper (T, Y)-ideal is contained in a maximal such, and each maximal proper (T, Y)-ideal is open.

<u>Proof</u>. If (P, Q) is a proper (T, Y)-ideal and $(s,x) \in S\backslash P \times X\backslash Q$, then $(P, Q) \leq (M(S\backslash s), M(S\backslash s, X\backslash x))$, an open proper (T, Y)-ideal which proves that the partially ordered collection $(\mathcal{B}, \leq)$ of open proper (T, Y)-ideals containing (P, Q) is nonvoid. Let (K, J) be the supremum of a maximal chain $C = \{(C_\lambda, D_\lambda) \mid \lambda \in \Lambda\}$ in $\mathcal{B}$, i.e. $(K, J) = \bigvee_{\lambda \in \Lambda} (C_\lambda, D_\lambda)$; (K, J) is seen to be an open (T, Y)-ideal.

If (K, J) is not proper then either $K = S$ or $J = X$, which implies, via the compactness of S and X, that S or X, as the case may be, is the union of finitely many open proper sets $C_{\lambda_1}, \ldots, C_{\lambda_n}$ (respectively $D_{\lambda_1}, \ldots, D_{\lambda_n}$). But the inputs as well as the state spaces are linearly ordered since C is, hence the union is just one of the C_{λ_i} (respectively, D_{λ_i}), which is absurd. Hence (K, J) is a proper ideal, and is clearly a maximal such. That (K, J) is open follows from lemma 2.7 via the equation $(K, J) = (M(S\backslash s), M(S\backslash s, X\backslash x))$ which holds for any $(s,x) \in S\backslash K \times X\backslash J$.

Proposition 2.3 is a description of the relation between the minimal ideal of S and the minimal (T, Y)-ideal of (S, X), viz, the existence of either implied the existence of the other, and the former is necessarily the input semigroup of the latter. Now in the case of maximal proper ideals, the situation is more complicated, in part because the existence of a maximal proper ideal does not imply its uniqueness, as is well-known from semigroup theory. Hence in light of 2.3, we may ask two questions:

(1) If K is a maximal proper T-ideal of S, when does there exist some $J \subseteq X$ such that (K, J) is a maximal proper (T, Y)-ideal?

(2) Conversely, if (K, J) is a maximal proper (T, Y)-ideal, when is K a maximal proper T-ideal of S?

Proposition 2.9 answers question (1); an answer to (2) is not known but lemma 2.9 sheds some light on the matter. For convenience we adopt a term used by Paalman-de Miranda [17]

and say that an act (S, X) has the <u>maximal property</u> relative to a subact (T, Y) if each proper (T, Y)-ideal is contained in a maximal such.

<u>Proposition 2.9</u>. If (S, X) has the maximal property relative to a subact (T, Y) and K is a maximal proper T-ideal of S such that $KY \neq X$ then there is some $J \subseteq X$ such that (K, J) is a maximal proper (T, Y)-ideal.

<u>Proof</u>. The hypotheses on K together with the trivial inequality $KY \cup T(KY) \subseteq KY$ imply that (K, KY) is a proper (T, Y)-ideal and hence, by the maximal property, is contained in some maximal proper (T, Y)-ideal (P, J). But then $K \subseteq P$ so the maximality if K implies $K = P$, proving the assertion.

<u>Lemma 2.10</u>. If (K, J) is a maximal proper (T, Y)-ideal and K' is a maximal proper T-ideal of S such that $K \subseteq K'$ then

(i) $K = K' \Longleftrightarrow K' \subseteq JY^{[-1]}$ and

(ii) $K = JY^{[-1]}$ or $JY^{[-1]} = S$.

<u>Proof</u>. Evidently if (K, J) is a (T, Y)-ideal then $K' = K \subseteq JY^{[-1]}$. Conversely if $K' \subseteq JY^{[-1]}$ then (K', J) is a proper (T, Y)-ideal which contains the maximal (T, Y)-ideal (K, J). Hence $K = K'$. For the second assertion we merely note that since $JY^{[-1]}$ is a T-ideal and $(JY^{[-1]})Y \cup TJ \subseteq J$, that $(JY^{[-1]}, J)$ is a (T, Y)-ideal; evidently $K \subseteq JY^{[-1]}$ so if $K \neq JY^{[-1]}$ the maximality of (K, J) implies $JY^{[-1]} = S$.

<u>Theorem 2.11</u>. Let (T, Y) be a closed subact of a compact act (S, X) and denote by E the set of idempotents of T. If (K, J) is a maximal proper (T, Y) ideal such that $E \subseteq K$, then $TY \subseteq J$.

Proof. Suppose to the contrary that for some $(t,y) \in T \times Y$, $ty \in TY \backslash J$. Necessarily $y \notin J$. It is clear that $(K, J \cup Ty)$ is a (T, Y)-ideal and that J is a proper subset of $J \cup Ty$. Hence the maximality of (K, J) implies $J \cup Ty = X$, and we may conclude from this that $y \in Ty$, i.e. $y = sy$ for some $s \in T$. Since the set $yy^{(-1)} = \{p \in S | y = py\}$ is closed and contains s^n for all $n \geq 1$ then it contains the compact subsemigroup $\Gamma(s) = \{s^n | n \geq 1\}^*$ and hence contains the idempotent of $\Gamma(s)$, i.e. $y = ey$, where $e^2 = e \in \Gamma(s)$. But $ey \in EY \subseteq KY \subseteq J$ by hypothesis so $y \in J$, which is absurd.

3. CATEGORICALITIES

We wish to prove a representation theorem which generalizes a result of Bednarek and Wallace [5] on inverse limits of finite machines. The generalization is most easily seen when it is couched in the language of categories and it is for this reason that we digress to include the necessary concepts from the theory of categories. A systematic exposition of this theory may be found in Mitchell [15] and a more compact exposition may be found in Freyd [11]. The reader who is familiar with the definition of a category may omit section 3.1.

3.1 Categories

A category $\mathcal{C}$ is an entity consisting of two classes $\mathcal{O}$ and $\mathcal{M}$, not necessarily sets, with $\mathcal{M}$ a disjoint union of the form $\bigcup_{(A,B)\in\mathcal{M}\times\mathcal{M}} \mathrm{Hom}_{\mathcal{C}}[A, B]$ satisfying the following axioms.

(1) $\mathrm{Hom}_{\mathcal{C}}[A, B]$ is a set for each pair $(A, B) \in \mathcal{O} \times \mathcal{O}$.

(2) For each triple $(A, B, C) \in \mathcal{O} \times \mathcal{O} \times \mathcal{O}$ there is a function called composition from $\mathrm{Hom}_{\mathcal{C}}[B, C] \times \mathrm{Hom}_{\mathcal{C}}(A, B)$ into $\mathrm{Hom}_{\mathcal{C}}[A, C]$ whose value at each point (α,β) is denoted by $\beta\alpha$, and which satisfies

(i) $(\gamma\beta)\alpha = \gamma(\beta\alpha)$ whenever both sides of this equation are defined;

(ii) For each $A \in \mathcal{O}$ there is an element $1_A \in \mathcal{M}$ $\mathrm{Hom}_{\mathcal{C}}(A, A)$ satisfying $1_A\alpha = \alpha$ and $\beta 1_A = \beta$ whenever the composition of $(1_A, \alpha)$ or $(\beta, 1_B)$ is defined.

The members of $\mathcal{O}$ are called objects of $\mathcal{C}$ and the members of $\mathcal{M}$ are called morphisms of $\mathcal{C}$. When there is no ambiguity we write Hom [A, B] instead of $\mathrm{Hom}_{\mathcal{C}}[A, B]$. If $\alpha \in \mathrm{Hom}\ [A, B]$ we call A the domain and B the codomain or range object of α, we write $A = \mathrm{dom}\ \alpha$ and $B = \mathrm{codom}\ \alpha$. The fact that $\alpha \in \mathrm{Hom}\ [A, B]$ is represented schematically by $A \xrightarrow{\alpha} B$ or $\alpha: A \longrightarrow B$.

It is easy to see that the morphism 1_A is unique for each A and hence we may unambiguously call it the identity morphism of A.

Certain types of morphisms are distinguished in a category. We define some of these next. A morphism α

of a category $\mathcal{C}$ is called a monomorphism of $\mathcal{C}$ (or is said to be monic) if whenever $\alpha x = \alpha y$ then necessarily $x = y$; it is an epimorphism of $\mathcal{C}$ (or is epic) of $x\alpha = y\alpha$ implies $x = y$; it is an isomorphism of $\mathcal{C}$ if there is a morphism β of the category such that $\beta\colon X \to A = \operatorname{dom} \alpha$, $\alpha\beta = 1_X$ and $\beta\alpha = 1_A$. We remark that an isomorphism is always both monic and epic but the converse need not hold.

A category $\mathcal{C}$ is a subcategory of a category $\mathbf{C}$ if each object of $\mathcal{C}$ is an object of $\mathbf{C}$ and if $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is a subset of $\mathrm{Hom}_{\mathbf{C}}(A, B)$ for each pair (A, B) of objects of $\mathcal{C}$. One must notice that the property of being epic or monic depends on the category being considered. It can be shown, e.g. that epimorphisms in the category of topological spaces (See example (ii) below) are the continuous surjections while in the subcategory of Hausdorff spaces the epimorphisms are the dense maps (a map $f\colon X \to Y$ is dense if $f(X)$ is a dense subset of Y), while in the sub-subcategory of compact Hausdorff spaces the epimorphisms are again the continuous surjections.

Examples of the categories abound:

(i) The category $\mathcal{S}$ of sets in which the objects are sets and α is in $\mathrm{Hom}_{\mathcal{S}}[A, B]$ if and only if α is a function from A into B. We must, in order to verify that $\mathcal{S}$ is a category, distinguish between $\alpha\colon X \to Y$ and $\hat{\alpha}\colon X \to Z$ in the case $Y \subseteq Z$ and $\alpha(x) = \hat{\alpha}(x)$ for each $x \in X$. For otherwise the class of morphisms is not a disjoint union of sets of morphisms. Such distinction will be assumed and rarely - if ever again - made explicit. Note that $\alpha\colon X \to Y$ is not

necessarily onto, of course.

(ii) The category G of topological spaces has the class of all topological spaces for objects and all continuous functions for morphisms. (The composition of continuous functions is again continuous, so the verification of the axioms 1 and 2 is routine.)

(iii) The category of discrete semigroups has for objects all discrete semigroups (i.e. semigroups S endowed with the discrete topology) and all homomorphisms for morphisms. The composition of homomorphisms is a homomorphism.

(iv) The category of topological semigroups or more simply of semigroups has all semigroups for objects (recall that the continuity of the semigroup operation is included in the definition of semigroup; see page 3) and all continuous semigroup homomorphisms for morphisms. Again continuity is part of the definition of semigroup homomorphism, so this category will be referred to simply as the category of semigroups.

Notice that the category of discrete semigroups is a subcategory of the category of semigroups, and that the class of compact Hausdorff spaces determines a subcateogry of the category of topological spaces; in this subcategory the monomorphisms are just the continuous injections, the epimorphisms are the continuous surjections and the isomorphisms are the bicontinuous bijections, i.e. homeomorphisms. It is also the case that the class of compact semigroups determine a subcategory of the category of semigroups. Here the monomorphisms, epimorphisms and isomorphisms are, respectively,

the one-to-one homomorphisms, surjective homomorphisms and the iseomorphisms (an iseomorphism is both a homeomorphism and an isomorphism). We remark that the category of discrete semigroups has an epimorphism which is not onto, namely the inclusion map of the natural numbers (zero included) into the integers, with addition the semigroup operation in both cases. The result follows from the fact that any homomorphism of the integers is determined by its restriction to the nonnegative integers. A characterization of the epimorphisms in the category of semigroups is not known to this writer. A topological example of this phenomenon is furnished by the inclusion map of the rationals into the reals (usual topologies) which is an epimorphism in the category of Hausdorff spaces and continuous functions.

An exposition of category theory may be found in the books by Mitchell [15] and Freyd [11] to which the reader is directed for a further knowledge of the theory.

> Proposition 3.1. The class of all acts and the class of all homomorphisms between acts are respectively the object class and morphism class of a category.

Proof. We remark first that an act is not completely specified until the sets S and X, together with a semigroup structure on S and an action of S on X, are given. So a homomorphism $(f,g)\colon (S, X) \to (T, Y)$ is not completely specified until the semigroup structures on S and T as well as the two actions are given. It is now seen to be the case that the class of homomorphisms is a disjoint union as

required by the definition of category. It is clear that the collection of homomorphisms between any two acts is a set, so axiom 1 is satisfied. From 1.5 (i) we see that part (i) of axiom 2 is satisfied if we define the composition $(f,g) \circ (\bar{f},\bar{g})$ of two homomorphisms to be $(f\bar{f},g\bar{g})$ whenever this makes sense. We may take the pair $(1_S, 1_X)$ of identity functions to satisfy part (ii) of axiom 2.

In the sequel this category will be called C for the sake of brevity. It will be convenient also to distinguish the subcategory C_o whose objects are compact acts and whose morphisms are those of C, i.e. (f,g) is to be a morphism of C_o iff $(f,g)\colon (S, X) \to (T, Y)$ and (S, X) and (T, Y) are both compact acts.

<u>Proposition 3.2</u>. Suppose $(f,g)\colon (S, X) \to (T, Y)$ is a homomorphism. Then (f,g) is an isomorphism in C iff f is an iseomorphism and g is a homeomorphism.

<u>Proof</u>. If (f,g) is an isomorphism then for some $(\bar{f},\bar{g})$ we have

$$(1_S, 1_X) = (\bar{f},\bar{g}) \circ (f,g) = (\bar{f}f,\bar{g}g) \tag{1}$$

and

$$(1_T, 1_Y) = (f,g) \circ (\bar{f},\bar{g}) = (f\bar{f},\bar{g}g) \tag{2}$$

so that f and g are invertible functions, from which fact the necessity follows. If, conversely, (f,g) is a homomorphism and f is an iseomorphism and g is a homeomorphism then the pair (f^{-1},g^{-1}) is a morphism of the category and satisfies equations (1) and (2) which proves that (f,g) is an isomorphism of C.

No such characterization of the monomorphisms or the epimorphisms of C is known.

We proceed next to investigate the existence in C of certain categorical constructs as prolegomena to a study of compact totally disconnected acts.

3.2 Products of Acts

Let $\mathcal{C}$ be a category and $\{A_\lambda : \lambda \in \Lambda\}$ a set of objects of $\mathcal{C}$. The product of the family is such a pair $(A, \{\pi_\lambda : \lambda \in \Lambda\})$ that A is an object of $\mathcal{C}$, $\pi_\lambda : A \to A_\lambda$ for each $\lambda \in \Lambda$ and such that if B is any object in $\mathcal{C}$ and $f_\lambda : B \to A_\lambda$ is given for each $\lambda \in \Lambda$ then there exists a unique $f: B \to A$, called the product morphism, such that $f_\lambda = \pi_\lambda f$ for each $\lambda \in \Lambda$, i.e. the diagram below commutes for each $\lambda \in \Lambda$:

$$\begin{tikzcd} B \arrow[rr, dashed, "f"] \arrow[dr, "f_\lambda"'] & & A \arrow[dl, "\pi_\lambda"] \\ & A_\lambda & \end{tikzcd}$$

Now if $(A, \{\pi_\lambda : \lambda \in \Lambda\})$, $(A', \{\rho_\lambda : \lambda \in \Lambda\})$ are products of the family $\{A_\lambda : \lambda \in \Lambda\}$ then two applications of the definition yield unique morphisms $f_1 : A \to A'$ and $f_2 : A' \to A$ such that $\pi_\lambda = \rho_\lambda f_1$ and $\rho_\lambda = \pi_\lambda f_2$ for each λ, so that $\pi_\lambda = \pi_\lambda f_2 f_1$. Now $\pi_\lambda = \pi_\lambda 1_A$, and the uniqueness of the product morphism allows us to conclude $f_2 f_1 = 1_A$. Similarly one shows that $f_1 f_2 = 1_{A'}$ which proves that A and A' are isomorphic. These results are well-known and it is for this reason that they are not stated as propositions here. We do, however, have the following result.

Proposition 3.3. The direct product of acts defined in section 1.1 is the product in C.

Proof. Let $\{(S_\lambda, X_\lambda) | \lambda \in \Lambda\}$ be any collection of acts. The direct product, as defined in section 1.1, of this family is the act $(\prod_{\lambda\in\Lambda} S_\lambda, \prod_{\lambda\in\Lambda} X_\lambda)$ where the action is coordinatewise, i.e. $(sx)_\lambda = s_\lambda x_\lambda$ for each $s \in S = \prod_{\lambda\in\Lambda} S_\lambda$, each $x \in X = \prod_{\lambda\in\Lambda} X_\lambda$ and each $\lambda \in \Lambda$. Now suppose (T, Y) is an act and $\{(f_\lambda, g_\lambda): (T, Y) \to (S_\lambda, X_\lambda) | \lambda \in \Lambda\}$ is a family of homomorphisms. Define $f: T \to S$ and $g: Y \to X$ by $(f(t))_\lambda = f_\lambda(t)$ and $(g(y))_\lambda = g_\lambda(y)$ for each $\lambda \in \Lambda$, i.e. $\rho_\lambda f = f_\lambda$ and $\sigma_\lambda g = g_\lambda$ for each $\lambda \in \Lambda$ where ρ_λ and σ_λ are the canonical projections of S and X onto S_λ and T_λ respectively. Now for each $\lambda \in \Lambda$, $(\rho_\lambda, \sigma_\lambda)$ is a homomorphism from the way the action of S on X is defined, namely $\sigma_\lambda(sx) = \rho_\lambda(s)\sigma_\lambda(x)$ for all $s \in S$, all $x \in X$ and all $\lambda \in \Lambda$. Then it follows that (f,g) is a homomorphism, for if $t, s \in S$ and $x \in X$ then

$\rho_\lambda f(ts) = f_\lambda(ts) = f_\lambda(t)f_\lambda(s) = \rho_\lambda(f(t))\rho_\lambda(f(s)) = \rho_\lambda(f(t)f(s))$,

i.e. $f(ts) = f(t)f(s)$ and

$\sigma_\lambda g(ts) = g_\lambda(ts) = f_\lambda(t)g_\lambda(s) = \rho_\lambda(f(t))\sigma_\lambda(g(s)) = \sigma_\lambda f(t)g(s)$,

i.e. $f(ts) = f(t)g(s)$.

It is elementary that f and g so defined are continuous and are unique with respect to making the diagrams below commute for each $\lambda \in \Lambda$.

$$\begin{tikzcd} T \arrow[rr, dashed, "f"] \arrow[dr, "f_\lambda"'] & & S \arrow[dl, "\sigma_\lambda"] \\ & S_\lambda & \end{tikzcd} \qquad \begin{tikzcd} Y \arrow[rr, "g"] \arrow[dr, "g_\lambda"'] & & X \arrow[dl, "\sigma_\lambda"] \\ & X_\lambda & \end{tikzcd}$$

Hence (f,g) is the unique homomorphism making the diagram below commute.

$$
\begin{tikzcd}
(T,\ Y) \arrow[rr, "{(f,g)}"] \arrow[dr, "{(f_\lambda, g_\lambda)}"'] & & (S,\ X) \arrow[dl, "{(\rho_\lambda, \sigma_\lambda)}"] \\
& (S_\lambda,\ X_\lambda) &
\end{tikzcd}
$$

3.3 Coproducts and Free Acts

The definition of coproduct or direct sum is dual to that of product in the sense that one is obtainable from the other by "reversing all the arrows" in either definition. We will defer exposing the definition however until we have discussed free acts [7], for the existence of the latter is needed to prove the existence of the former.

In the category of discrete semigroups, a free semigroup on a set A is a pair $(\omega, F(A))$ with ω such a function and $F(A)$ such a semigroup that, if S is any semigroup then any function $f: A \to S$ extends uniquely to a homomorphism $\bar{f}: F(A) \to S$ so that the diagram below commutes.

$$\begin{tikzcd} A \arrow[rr, "\omega"] \arrow[dr, "f"'] & & F(A) \arrow[dl, dashed, "\bar{f}"] \\ & S & \end{tikzcd}$$

It is well-known that any two free semigroups on A are isomorphic and that a free semigroup exists on every set: let $F(A) = \bigcup_{n=1}^{\infty} A^n$ with the operation given by

$$(a_1, \ldots a_p)(b_1, \ldots b_q) = (a_1, \ldots a_p, b_1, \ldots b_q)$$

for all $a_1, \ldots a_p, b_1, \ldots b_q \in A$ all positive integers p and q; $\omega: A \to F(A)$ is $\omega(a) = (a)$. This particular representation F(A) is called the canonical free semigroup on A, and leads us to a representation of the notion of free act.

Let now (A, B) be any ordered pair of sets. The <u>free act on (A, B)</u> is an act (F(A), F(A, B)) together with a pair of functions ω_1: A → F(A) and ω_2: B → F(A, B) such that if (S, X) is any act and (f,g): (A, B) → (S, X) is any pair of functions, then there is a unique homomorphism $(\bar{f},\bar{g})$ making the diagram below commutative.

$$\begin{tikzcd}
(A, B) \arrow[rr, "{(\omega_1,\omega_2)}"] \arrow[dr, "{(f,g)}"'] & & (F(A), F(A, B)) \arrow[dl, dashed, "{(\bar{f},\bar{g})}"] \\
& (S, X) &
\end{tikzcd}$$

<u>Proposition 3.4</u>. In the category of discrete acts, a free act exists on every pair of sets.

<u>Proof</u>. Let F(A, B) = (F(A) × B) ∪ B where F(A) is the canonical free semigroup on A. We define a function F(A) × F(A, B) $\overset{*}{\to}$ F(A, B) as follows: If s ∈ F(A), s = $(a_1,\ldots a_p)$ say and b ∈ B then s * b = $((a_1,\ldots a_p),b)$. If s is as above and x = $((x_1,\ldots x_q),b) \in A^q \times B$ then s * x = $((a_1,\ldots a_p,x_1,\ldots x_q),b)$. It is easy to see that * is an action. Take (i,j): (A, B) → (F(A), F(A, B)) to be i(a) = a and j(b) = b for each a ∈ A and each b ∈ B.

Suppose now that (f,g): (A, B) → (S, X) is any pair of functions from (A, B) into (S, X). Since F(A) is the free semigroup on A, there is exactly one homomorphism $\bar{f}$: F(A) → S such that $\bar{f}i$ = f. Define $\bar{g}$: F(A, B) → X by $\bar{g}(b) = g(b)$ for b ∈ B and $\bar{g}((x_1,\ldots x_q),b) = \bar{f}(x_1,\ldots x_q)g(b)$ for all $((x_1,\ldots x_q),b) \in F(A) \times B$. It can be seen that $(\bar{f},\bar{g})$ is a homomorphism. In fact, if $(a_1,\ldots a_p) \in F(A)$ and $((b_1,\ldots b_q),b) \in F(A) \times B$ then $\bar{f}(a_1,\ldots a_p)\bar{g}((b_1,\ldots b_q),b) =$
$\bar{f}(a_1,\ldots a_p)(\bar{f}(b_1,\ldots b_q)g(b)) = \bar{f}(a_1,\ldots a_p,b_1,\ldots b_q)g(b) =$
$\bar{g}((a_1,\ldots a_p,b_1,\ldots b_q),b) = \bar{g}((a_1,\ldots a_q)*((b_1,\ldots b_q),b))$;

also if $b \in B$ and $(a_1,\ldots a_p) \in F(A)$ then $\overline{f}(a_1,\ldots a_p)\overline{g}(b) = \overline{f}(a_1,\ldots a_p)g(b) = \overline{g}((a_1,\ldots a_p),b) = \overline{g}(a_1,\ldots a_p) * b)$. Hence $(\overline{f},\overline{g})$ is a homomorphism and $(\overline{f},\overline{g}) \circ (i,j) = (f,g)$, proving the proposition.

Evidently if $A = B$ then $F(A, B) = F(A)$, and the free act on (A, A) is the free semigroup $F(A)$ acting on itself. One can prove, just as for semigroups, that every act (S, X) is isomorphic to a quotient of a free act; the free act in question is just $(F(S), F(S, X))$ and it follows that (S, X) is the image of this free act and hence (S, X) is isomorphic to a quotient of $(F(S), F(S, X))$. We can now discuss the notion of coproduct in the category of discrete acts.

If $\{A_\lambda | \lambda \in \Lambda\}$ is any family of objects in a category $\mathcal{C}$, the <u>coproduct</u> of the family is any object A together with a family of morphisms $\{i_\lambda : A_\lambda \to A | \lambda \in \Lambda\}$ having the property that if X is any object in $\mathcal{C}$ and $f_\lambda : A_\lambda \to X$ is a morphism for each $\lambda \in \Lambda$, then there is exactly one morphism f making the diagram below commute for each $\lambda \in \Lambda$:

$$\begin{tikzcd} X & & A \arrow[ll, dashed, "f"'] \\ & A_\lambda \arrow[ul, "f_\lambda"] \arrow[ur, "i_\lambda"'] & \end{tikzcd}$$

The uniqueness of f implies that any two coproducts of $\{A_\lambda | \lambda \in \Lambda\}$ are isomorphic. Proof of this assertion is similar to that of the similar assertion for products and is omitted. We use $\coprod_{\lambda \in \Lambda} A_\lambda$ to denote any coproduct of $\{A_\lambda | \lambda \in \Lambda\}$.

<u>Proposition 3.5</u>. Coproducts exist in the category of discrete acts.

<u>Proof</u>. Let $\{(S_\lambda, X_\lambda) \mid \lambda \in \Lambda\}$ be any family of discrete acts; it is no loss of generality to suppose that $S_\lambda \cap S_\mu = \emptyset = X_\lambda \cap X_\mu$ whenever $\lambda \neq \mu$, so we do make this supposition. We also define $S = \bigcup_{\lambda\in\Lambda} S_\lambda$, $X = \bigcup_{\lambda\in\Lambda} X_\lambda$, and define $p_\lambda\colon S_\lambda \to S$ by $p_\lambda(s) = s$ for all $s \in S_\lambda$, $q_\lambda\colon X_\lambda \to X$ by $q_\lambda(x) = x$ for all $x \in X_\lambda$. If now (T, Y) is a discrete act and $(f_\lambda, g_\lambda)\colon (S_\lambda, X_\lambda) \to (T, Y)$ is a homomorphism for each $\lambda \in \Lambda$, then define $\hat{f}\colon S \to T$ and $\hat{g}\colon X \to Y$ by

$\hat{f}(s) = f_\lambda(s)$ if $s \in S_\lambda$ and

$\hat{g}(x) = g_\lambda(x)$ if $x \in X_\lambda$.

These functions are well-defined since the S_λ's and the X_λ's are both pairwise disjoint families. Letting $(\bar{i},\bar{j})$ be the natural injection of (S, X) into $(F(S), F(S, X))$ we have that $(\hat{f},\hat{g})$ extends to a unique homomorphism $(\bar{f},\bar{g})\colon (F(S), F(S, X)) \to (T, Y)$ by proposition 3.4.

Now define the sets $F_\circ$ and $E_\circ$ as follows: $F_\circ = \{(z,w) \in F(S) \times F(S) \mid \exists\ \lambda \in \Lambda \cdot\ (x,y) \in S_\lambda \times S_\lambda \cdot \bar{i}p_\lambda(xy) = z$ and $\bar{i}p_\lambda(x)\bar{i}p_\lambda(y) = w\}$. $E_\circ = \{(a,b) \in F(S, X) \times F(S, X) \mid \exists\ \lambda \in \Lambda \cdot\ (s,x) \in S_\lambda \times X_\lambda \cdot\ni\cdot\ \bar{j}q_\lambda(sx) = a$ and $\bar{i}p_\lambda(s) \cdot \bar{j}q_\lambda(x) = b\}$. Since $F_\circ \subseteq F(S) \times F(S)$ and $E_\circ \subseteq F(S, X) \times F(S, X)$, the collection of all admissible pairs $(P, Q) \geq (F_\circ, E_\circ)$ is not empty; let (F, E) denote the infimum of all such admissible pairs and let (p,q) be the canonical homomorphism from $(F(S), F(S, X))$ to $(F(S)/F, F(S, X)/E)$. Referring to the commutative diagram

below, we verify next that $E \subseteq \operatorname{Ker} \bar{g}$.

$$\begin{tikzcd}
 & (S, X) \arrow[r, "(\bar{i},\bar{j})"] \arrow[dd, "(\hat{f},\hat{g})"] & (F(S), F(S, X)) \arrow[ddl, dashed, "(\bar{f},\bar{g})"'] \arrow[ddr, "(p,q)"] & \\
(S_\lambda, X_\lambda) \arrow[ur, "(p_\lambda,q_\lambda)"] \arrow[dr, "(f_\lambda,g_\lambda)"'] & & & \\
 & (T, Y) & & (F(S)/F, F(S, X)/E)
\end{tikzcd}$$

Indeed if $(a,b) \in E$ then for some $\lambda \in \Lambda$ there is some $(s,x) \in S_\lambda \times X_\lambda$ such that

$$a = \bar{j}q_\lambda(sx) \text{ and } b = \bar{i}p_\lambda(s) \cdot \bar{j}q_\lambda(x).$$

Hence $\bar{g}(a) = \bar{g}(\bar{j}q_\lambda(sx)) = (\overline{g}\bar{j})q_\lambda(sx) = \hat{g}q_\lambda(sx) = g_\lambda(sx) =$
$f_\lambda(s) \cdot g_\lambda(x) = \hat{f}p_\lambda(s) \cdot \hat{g}q_\lambda(x) = (\bar{f}\bar{i})p_\lambda(s) \cdot (\bar{g}\bar{j})q_\lambda(x) =$
$\bar{f}(\bar{i}p_\lambda(s)) \cdot \bar{g}(\bar{j}q_\lambda(x)) = \bar{g}(\bar{i}p_\lambda(s) \cdot \bar{j}q_\lambda(x)) = \bar{g}(b)$,

i.e. $E \subseteq \operatorname{Ker} g$.

It follows <u>mutatis</u> <u>mutandis</u> that $F \subseteq \operatorname{Ker} \bar{f}$, so that $(F, E) \leq K(\bar{f},\bar{g})$; since p and q are surjections, the hypothesis of 1.2′ is satisfied, allowing one to conclude the existence of a unique homomorphism $(f,g)\colon (F(S)/F, F(S, X)/E) \to (T, Y)$ satisfying

$$(fp,gq) = (\bar{f},\bar{g}).$$

In summary we have shown, letting $(i,j) = (p\bar{i}p_\lambda, q\bar{j}q_\lambda)$, that there is a unique (f,g) making the diagram below analytic for each $\lambda \in \Lambda$, proving that $(F(S)/F, F(S, X)/E) = \coprod_{\lambda \in \Lambda} (S_\lambda, X_\lambda)$.

$$\begin{tikzcd}
(S_\lambda, X_\lambda) \arrow[rr, "(i,j)"] \arrow[dr, "(f_\lambda,g_\lambda)"'] & & (F(S)/F, F(S, X)/E) \arrow[dl, dashed, "(f,g)"] \\
 & (T, Y) &
\end{tikzcd}$$

3.4 Inverse Limits of Compact, Totally Disconnected Acts

We wish to formulate a sharper version of a result of Bednarek and Wallace (Theorem 1, [5]) concerning the state space of a compact totally disconnected act. It is necessary first to define the notion of inverse limit in a category.

We recall that a quasi-ordered set $(\Lambda,\leq)$ is a set Λ with a relation $\leq$ on Λ which is reflexive and transitive. If $\lambda \leq \mu$ it is sometimes convenient to write $\mu \geq \lambda$. A quasi-ordered set $(\Lambda,\leq)$ is called <u>directed</u> if for each $(\lambda,\mu) \in \Lambda \times \Lambda$ there is some $\nu \in \Lambda \ni \lambda \leq \nu$ and $\mu \leq \nu$.

An <u>inverse system</u> $(A_\lambda, p^\lambda_\mu, \Lambda, \leq)$ in a category $\mathcal{C}$ is a family of objects $\{A_\lambda | \lambda \in \Lambda\}$ indexed by a directed set $(\Lambda,\leq)$ together with a family $\{p^\lambda_\mu | \lambda \geq \mu\}$ of morphisms, $p^\lambda_\mu\colon A_\lambda \to A_\mu$ such that $p^\lambda_\lambda = 1_{A_\lambda}$ and if $\lambda \geq \mu \geq \nu$, then the diagram below commutes:

$$\begin{tikzcd} A_\lambda \arrow[r, "p^\lambda_\mu"] \arrow[dr, "p^\lambda_\nu"'] & A_\mu \arrow[d, "p^\mu_\nu"] \\ & A_\nu \end{tikzcd}$$

The <u>inverse</u> <u>limit</u> in $\mathcal{C}$ of an inverse system $(A_\lambda, p^\lambda_\mu, \Lambda, \leq)$ is an object L of $\mathcal{C}$ together with a collection $\{\pi_\lambda\colon L \to A_\lambda | \lambda \in \Lambda\}$ of morphisms of $\mathcal{C}$ such that if X is any object of $\mathcal{C}$ and $\{f_\lambda\colon X \to A_\lambda | \lambda \in \Lambda\}$ is any family of morphisms of $\mathcal{C}$ such that $p^\lambda_\mu f_\lambda = f_\mu$ whenever $\lambda \geq \mu$, then there is

exactly one morphism f making each diagram with $\lambda \geq \mu$ commute:

$$\begin{tikzcd} X \arrow[rr, dashed, "f"] \arrow[dr, "f_\lambda"'] & & L \arrow[dl, "\pi_\lambda"] \\ & A_\lambda & \end{tikzcd}$$

The uniqueness of f implies that the inverse limit of any inverse system is unique up to isomorphism. The proof of this assertion is quite similar to the proof of uniqueness of the direct product and is omitted. It is occasionally convenient to write $\varprojlim A_\lambda$ for the inverse limit of a system $(A_\lambda, p_\mu^\lambda, \Lambda, \leq)$. If $(A_\lambda, p_\mu^\lambda, \Lambda, \leq)$ is an inverse system in the category of compact Hausdorff spaces then $\varprojlim A_\lambda =$ $\{x \in \prod_{\lambda \in \Lambda} A_\lambda \mid \lambda \geq \mu \Longrightarrow p_\mu^\lambda \, \pi_\lambda(x) = \pi_\mu(x)\}$ and is a closed, non-empty subset of $\prod_{\lambda \in \Lambda} A_\lambda$. These assertions are proved in Eilenberg and Steenrod, Chapter VIII [10].

If S is a compact semigroup then the class of all compact acts (S, X) are the objects of a category $\mathcal{C}(S)$, the morphisms of which are homomorphisms of the form $(1_S, g)$. The result of Bednarek-Wallace alluded to above can be stated now as follows.

<u>Theorem 3.6</u>. (Bednarek-Wallace). A compact act (S, X) with X totally disconnected is the inverse limit in $\mathcal{C}(S)$ of acts (S, X_λ) having finite, discrete state spaces.

The techniques used to prove this theorem may be applied to prove the following result.

Theorem 3.7. If (S, X) is a compact, totally disconnected act (i.e. both S and X are compact and totally disconnected) then (S, X) is the inverse limit in the category of compact acts of finite discrete acts.

The theorem will be proved from the following sequence of lemmas. The references below are to Numakura [16] and Bednarek and Wallace [5]. We recall that for any set Y that the *diagonal* of Y is $\Delta_Y = \{(y,y) \mid y \in Y\}$; Δ_Y is closed in $Y \times Y$ if and only if Y is a Hausdorff space.

Lemma 3.8. (Numakura). If Y is a compact Hausdorff space and E is an open equivalence relation on Y then Y/E is finite and discrete.

Lemma 3.9. (Numakura). If Y is a compact, totally disconnected space and U is an open subset of Y containing Δ_Y, then there is an open equivalence relation E on Y contained in U. Moreover if Y is additionally a semigroup then there is an open congruence $E \subseteq U$.

It is well-known (See, e.g. [5]) that an open equivalence relation on a Hausdorff space is necessarily closed, i.e. clopen.

Lemma 3.10. (Bednarek-Wallace). If (S, X) is a compact act, X is totally disconnected and V is an open subset of $X \times X$ containing the diagonal of X, then there is such a clopen equivalence $E \subseteq V$ that (Δ_S, E) is an admissible pair.

Lemma 3.11. If (S, X) is a compact totally disconnected act and U and V are such open subsets of S and of X respectively that $\Delta_S \subseteq U$ and $\Delta_X \subseteq V$, then there is a clopen admissible pair $(F, E) \leq (U, V)$.

Proof. We may deduce from an application of 3.10 that there is a clopen equivalence relation $E \subseteq V$ such that $\Delta_S E \subseteq E$. Since E is in particular open then the continuity of the action implies via Wallace's theorem that there is an open set $U_\circ$ containing the compact set Δ_S which has the property that $U_\circ E \subseteq E$. An application of 3.9 to the semi-group S implies the existence of a clopen congruence $F \subseteq U_\circ \cap U$. Now $FE \subseteq U_\circ E \subseteq E$, proving that (F, E) is an admissible pair. Since $(F, E) \leq (U, V)$ the lemma is proved. We return now to the main result of this section.

Proof of 3.7. If (S, X) is a compact, totally disconnected act then 3.11 implies that the family $\mathcal{J}$ of all clopen admissible pairs $(F, E) \leq (S \times S, X \times X)$ is a nonvoid family; clearly $\mathcal{J}$ is a filter base relative to the partial order $\leq$ on $P(S) \times P(X)$ (i.e. if (F_1, E_1) and (F_2, E_2) are in $\mathcal{J}$ then there is some $(F_3, E_3) \leq (F_1, E_1) \wedge (F_2, E_2)$; we may take $F_3 = F_1 \cap F_2$ and $E_3 = E_1 \cap E_2$). Furthermore, $\wedge\mathcal{J} = (\Delta_S, \Delta_X)$. Indeed, if $(s,t) \in S \times S \backslash \Delta_S$ and $(x,y) \in X \times X \backslash \Delta_X$, then the regularity of $X \times X$ implies the existence of an open set $V \subseteq X \times X$ such that $\Delta_X \subseteq V$ and $(x,y) \notin V$. Similarly there is an open set U such that $\Delta_S \subseteq U$ and $(s,t) \notin U$. Then apply 3.11 to obtain $(F, E) \in \mathcal{J}$ with $(s,t) \notin F$ and $(x,y) \notin E$.

Now we index $\mathcal{J}$ by a suitable set Λ and, observing that $\mathcal{J}$ is quasi-ordered by $\leq$, define a relation $<$ on Λ by declaring $\lambda < \mu$ if and only if $(F_\mu, E_\mu) \leq (F_\lambda, E_\lambda)$. Since $\mathcal{J}$ is a filter base it follows that $(\Lambda,<)$ is a directed set. If we denote the canonical pair of quotient surjections of (S, X) upon $(S/F_\lambda, X/E_\lambda)$ by (p_λ,q_λ), then whenever $\lambda \geq \mu$ it follows from Sierpinski's theorem (1.2′) that there is a homomorphism $(f_\mu^\lambda,g_\mu^\lambda)$ making the diagram below commute.

$$\begin{tikzcd}
(S, X) \arrow[d, "(p_\lambda, q_\lambda)"] \arrow[dr, "(p_\mu, q_\mu)"] & \\
(S/F_\lambda, X/E_\lambda) \arrow[r, "(f_\mu^\lambda, g_\mu^\lambda)"] & (S/F_\mu, X/E_\mu).
\end{tikzcd}$$

Evidently $f_\nu^\lambda = f_\nu^\mu f_\mu^\lambda$, $g_\nu^\lambda = g_\nu^\mu g_\mu^\lambda$ if $\lambda \geq \mu \geq \nu$, and $(f_\lambda^\lambda,g_\lambda^\lambda) = (1_{S/F_\lambda}, 1_{X/E_\lambda})$ for all $\lambda \in \Lambda$. Now let $S_\infty =$ $\{s \in \prod_{\lambda\in\Lambda} S/F_\lambda \mid \lambda \geq \mu \Longrightarrow f_\mu^\lambda\, \pi_\lambda(s) = \pi_\mu(s)\}$, (i.e. S_∞ is the canonical representation of $\varprojlim S/F_\lambda$) and define X_∞ similarly. It is easy to check that (S_∞, X_∞), considered as a subact of $\prod_{\lambda\in\Lambda}(S/F_\lambda, X/E_\lambda)$, is the inverse limit in the category of compact acts of the inverse system $(S/F_\lambda, X/E_\lambda)$ of acts. Indeed if $(f_\lambda,g_\lambda)\colon (T, Y) \to (S/F_\lambda, X/E_\lambda)$ is given for any act (T, Y) and all $\lambda \in \Lambda$, and satisfies $(f_\mu^\lambda,g_\mu^\lambda)\circ(f_\lambda,g_\lambda) =$ (f_μ,g_λ) whenever $\lambda \geq \mu$, then there is a semigroup homomorphism $f\colon T \to S_\infty$ and map $g\colon Y \to X_\infty$ satisfying $\pi_\lambda f = f_\lambda$ and $\rho_\lambda g = g_\lambda$ for all λ, where π_λ and ρ_λ are the projection functions from $\prod_{\lambda\in\Lambda} S/F_\lambda$ onto S/F_λ and from $\prod_{\lambda\in\Lambda} X/E_\lambda$ onto

X/E_λ respectively. We show first that (S_∞, X_∞) is a subact of the product. If $\lambda \geq \mu$ and $(s,x) \in S_\infty \times X_\infty$ then $g^\lambda_\mu \rho_\lambda(sx) = g^\lambda_\mu(\pi_\lambda(s)\rho_\lambda(x)) = (f^\lambda_\mu \pi_\lambda(s))(g^\lambda_\mu \rho_\lambda(x)) = \pi_\mu(s)\rho_\mu(x) = \rho_\mu(sx)$, since $(f^\lambda_\mu, g^\lambda_\mu)$ and $(\pi_\lambda, \rho_\lambda)$ are all homomorphisms. Hence $S_\infty X_\infty \subseteq X_\infty$. To see that (f,g) is a homomorphism, let $t \in T$ and $y \in Y$. Then $\rho_\lambda g(ty) = g_\lambda(ty) = f_\lambda(t)g_\lambda(y) = \pi_\lambda f(t)\rho_\lambda g(y) = \rho_\lambda(f(t)g(y))$ for each $\lambda \in \Lambda$, i.e. $g(ty) = f(t)g(y)$. Hence we have proved the assertion that $(S_\infty, X_\infty) = \varprojlim (S/F_\lambda, X/F_\lambda)$. Now in particular there is a homomorphism $(\overline{f},\overline{g})\colon (S, X) \to (S_\infty, X_\infty)$ satisfying $\pi_\lambda \overline{f} = p_\lambda$ and $\rho_\lambda \overline{g} = q_\lambda$ for each $\lambda \in \Lambda$. Since $\wedge\mathcal{J} = (\Delta_S, \Delta_X)$, then $\overline{f}$ and $\overline{g}$ separate points of S and X respectively. It follows now that $\overline{f}$ and $\overline{g}$ are homeomorphisms, so that we may conclude from 3.2 that $(\overline{f},\overline{g})$ is an isomorphism; this observation concludes the proof of 3.7.

3.5 An Embedding Theorem

If (S, X) is such an act that $s(tx) = t(sx)$ for all $s,t \in S$ and all $x \in X$, we say that S acts commutatively on X; if $s(sx) = sx$ for each $s \in S$ and each $x \in X$, we say S acts idempotently on X; if S acts both commutatively and idempotently on X we call the action trellis-like, and say S acts trellisly on X [5]. The action of S on X is called effective if, for each pair of distinct elements s,t in S there is some $x \in X$ such that $sx \neq tx$. Notice that a semigroup may act on itself either commutatively or idempotently without being a commutative semigroup or a semigroup each of whose elements is idempotent. For example, if S is any Hausdorff space define $st = t$ for all $(s,t) \in S \times S$. Then S is a semigroup and $s(tx) = tx = x = sx = t(sx)$ but S is not commutative if it contains at least two elements; if next $t_0 \in S$ is fixed and we define $st = t_0$ for all $(s,t) \in S \times S$ then any $t \neq t_0$ is not idempotent, but for all $s,t,x \in S$ we have $t(tx) = t_0 = tx$. Of course, if S is commutative (respectively, idempotent) then S acts commutatively (respectively, idempotently) on any space on which it acts. However, if S acts effectively on X then S acts trellisly iff S is a semilattice. The proof is an immediate consequence of the fact that there is a canonical embedding Φ of S into X^X, the semigroup of all functions from X to itself, given by $[\Phi(s)](x) = sx$ for each $s \in S$ and each $x \in X$; Φ is evidently a one-to-one homomorphism.

Hence if S acts commutatively, $\Phi(S)$ is a commutative subsemigroup of X^X, i.e. S is commutative; a similar remark holds if "commutative" is everywhere replaced by "idempotent."

We recall that a <u>semilattice</u> is a commutative, idempotent semigroup (in the sense in which "commutative" and "idempotent" are usually used in algebra). By the preceding remark any action by a semilattice is necessarily trellis-like.

A <u>prime</u> right (left) <u>ideal</u> of an act is such a right (left) ideal (A, B) that if $tx \in B$ then either $t \in A$ or $x \in B$. A <u>prime</u> <u>ideal</u> is both a prime left and prime right ideal. The definition can be made for prime <u>relative</u> ideals, of course, but we will have need only of the notion in the "absolute" case. For any act (S, X), the <u>index</u> of an admissible pair (F, E) is an ordered pair (p,q) of cardinal numbers such that p is the cardinal number of S/F and q is the cardinal number of X/E.

In the paper "Finite Approximants of Compact Totally Disconnected Machines" [5], the following theorem was proved.

> <u>Theorem 3.12</u>. (Bednarek-Wallace) If (S, X) is a compact trellis-like act with $X = Sz$ for some $z \in X$, and X is totally disconnected, then there is an isomorphism $(1_T, \Phi)$ of (S, X) into an act (S, B) where B is a compact boolean lattice.

(Theorem 3.12 is only a part of theorem 2 of [5]; according to a theorem of Aczel and Wallace, [1], [23], such a state space X must support the structure of a semigroup and then the remainder of theorem 2 of [5] asserts that Φ is a semigroup homomorphism into the $\wedge$-semilattice

of the boolean lattice B.) We state next the principal theorem of this paper, and then a sequence of lemmas needed for its proof.

Theorem 3.13. If (S, X) is a compact, totally disconnected, effective trellis-like act such that neither card S nor card X is 1 and $x \in Sx$ for each $x \in X$ then (S, X) can be embedded in a direct product of acts (S_λ, X_λ) where, for each λ, (S_λ, X_λ) is one of the following two acts:

(i) (S_λ, X_λ) is the semilattice of order 2 acting on itself via its multiplication

(ii) S_λ is trivial and the action satisfies the identity $tx = x$ for each $x \in X_\lambda$.

The proof of 3.13 is deferred until the following lemmas have been proved; the first two parts of 3.14 are from [5], but the remaining results seem to be new.

Lemma 3.14. If S acts trellisly on X and $x \in Sx$ for each $x \in X$ then

(i) the relation $P = \{(x,y) | x \in Sy\}$ is a partial order on X, called the natural order on X

(ii) $\Delta_T E \subseteq E$ if $E = (X \backslash S^{(-1)}a \times X \backslash S^{(-1)}a) \cup (S^{(-1)}a \times S^{(-1)}a)$ for any element $a \in X$

(iii) letting $\psi\colon X \to X/E$ be the canonical surjection, then $\overline{P} = [\psi \times \psi](P)$ is a partial order on X/E.

We remark that in the proof of (ii) one uses the fact that $(S, X\ S^{(-1)}a)$ is a left ideal of (S, X). The proof of (iii) is immediate since $\overline{P}$ is the natural order on X/E, and the quotient act (S, X/E) itself satisfies the hypothesis of lemma 1.

Lemma 3.15. Let (S, X) be an act with such an element x in X that $S\backslash xX^{(-1)}$ is not empty. Then $S\backslash xX^{(-1)}$ is (i) a right ideal of S which is prime if S acts idempotently; (ii) an ideal if S acts commutatively

Proof (i). Suppose to the contrary that $t \in S\backslash xX^{(-1)}$, $s \in S$ and $ts \in xX^{(-1)}$. Then for some $x_0 \in X$ we have $x = (ts)x_0 = tsx_0$, from which it is concluded that $t \in xX^{(-1)}$, contrary to the supposition $t \notin xX^{(-1)}$. Hence $S\backslash xX^{(-1)}$ is a right ideal of S. If now p and q are in $xX^{(-1)}$ then $x \in pX \cap qX$, so that $x = py_1 = qy_2$ for some y_1, $y_2 \in X$; since the action is idempotent, $x = p(py_1) = p(qy_2) = (pq)y_2$, which implies that $pq \in xX^{(-1)}$ so that $xX^{(-1)}$ is a semigroup. But this means that $S\backslash xX^{(-1)}$ is prime.

(ii) If now S acts commutatively and $S\backslash xX^{(-1)}$ is not a left ideal then there is some $s \in S$ and some $t \in S\backslash xX^{(-1)}$ such that $st \in xX^{(-1)}$ which is to say $x = stx_0$ for some $x_0 \in X$. But $stx_0 = tsx_0$, so that $x \in tX$; this implies $t \in xX^{(-1)}$, which is absurd.

Lemma 3.16. If I is a prime ideal of a semigroup S or $I = \emptyset$ then $(I \times I) \cup (S\backslash I \times S\backslash I)$ is a congruence on S.

The proof is omitted. The following special case will be needed in the sequel.

Corollary 3.17. If (S, X) is a trellis-like act and $x \in X$ then $F = (S\backslash xX^{(-1)} \times X\backslash xX^{(-1)}) \cup (xX^{(-1)} \times xX^{(-1)})$ is a congruence on S.

We remark that $S\backslash xX^{(-1)}$ may be empty so that $F = S \times S$. For example if $X = S$ is a semilattice acting on itself via its multiplication and having a zero element x then $xX^{(-1)} = S$.

<u>Lemma 3.18</u>. Let (S, X) be a trellis-like act and $x \in X$. Then

(i) $(S\backslash xX^{(-1)})(S^{(-1)}x) \subseteq X\backslash S^{(-1)}x$

(ii) $S(X\backslash S^{(-1)}x) \subseteq X\backslash S^{(-1)}x$

(iii) $(xX^{(-1)})(S^{(-1)}x) \subseteq S^{(-1)}x$

<u>Proof</u>. (i) If $s \in S\backslash xX^{(-1)}$ then $x \in X\backslash sX$; if also $y \in S^{(-1)}x$ and $sy \in S^{(-1)}x$, then $x \in S(sy) = s(Sy) \subseteq sX$, a contradiction.

(ii) If $y \in X\backslash S^{(-1)}x$ and if for some $s \in S$, $sy \in S^{(-1)}x$ then $x \in Ssy \subseteq Sy$, i.e. $y \in S^{(-1)}x$, a contradiction.

(iii) If $s \in xX^{(-1)}$ and $y \in S^{(-1)}x$ then for some $x_1 \in X$ and some $s_1 \in S$ we have $x = sx_1 = s_1y$. Then $x = sx_1 = s(sx_1) = sx = s(s_1y) = s_1(sy) \in Ssy$, which is to say $sy \in S^{(-1)}x$, proving (iii).

<u>Corollary 3.19</u>. Let (S, X) be a trellis-like act and $x \in X$; take F as in 3.17 and E as in 3.14 (iii) with $a = x$; then (F, E) is an admissible pair.

<u>Proof</u>. The only thing not already verified is that $FE \subseteq E$, but this follows from the definitions of F and E and an application of 3.18.

In the sequel let $\underline{2} = \{0,1\}$ be the semilattice with $0 \leq 1$ and let $\underline{1} = \{1\}$ be a subsemilattice of $\underline{2}$. We define $(\underline{2},\underline{2})$ to be the action of $\underline{2}$ on $\underline{2}$ via semilattice multiplication:

	0	1
0	0	0
1	0	1

and $(\underline{1},\underline{2})$ is understood as a subact of $(\underline{2},\underline{2})$. Its multiplication table is

	0	1
1	0	1

<u>Lemma 3.20</u>. Let (S, X) be an effective trellis-like discrete act such that $a \in S^{(-1)}a$ for each $a \in X$, and suppose (s,x) and (t,y) are distinct points of $S \times X$. Then there is an admissible pair (F, E) on (S, X) of index $(1,2)$ or $(2,2)$ such that $(\phi(s), \psi(x)) \neq (\phi(t), \psi(y))$ where $(\phi,\psi)\colon (S, X) \to (S/F, X/E)$ is the canonical morphism. Furthermore, (i) if (F, E) has index $(1,2)$ then $(S/F, X/E)$ is isomorphic to $(\underline{1},\underline{2})$; (ii) if the index of (F, E) is $(2,2)$ then $(S/F, X/E)$ is isomorphic to $(\underline{2},\underline{2})$.

<u>Proof</u> of (i). If $(s,x) \neq (t,y)$ then either $x \neq y$, or $x = y$ but $s \neq t$. We consider first the case $x \neq y$ and we may suppose without loss of generality that $y \notin S^{(-1)}x$. (Indeed if $y \in S^{(-1)}x$ and $x \in S^{(-1)}y$ then $x \in Sy$ and $y \in Sx$, which implies via 3.14 (i) that $x = y$.) Apply 3.18 to the point x to obtain the admissible pair (F, E). Since (S, X) is discrete, $(S/F, X/E)$ is an act and since $y \in X\ S^{(-1)}x$ and $x \in S^{(-1)}x$ by hypothesis, then $\psi(x) \neq \psi(y)$. In this case the hypothesis of effectivity is not used, a fact of particular importance in the sequel. In case $x = y$

and $s \neq t$, then the effectivity hypothesis implies that $sx_0 \neq tx_0$ for some $x_0 \in X$. As before, it is no loss of generality to suppose that $tx_0 \notin S^{(-1)}sx_0$, and with this assumption we claim that $t \notin (sx_0)X^{(-1)}$. For otherwise $sx_0 \in tX$, and hence there is some $x_1 \in X$ such that $sx_0 = tx_1$, implying $sx_0 = tx_1 = t(tx_1) = t(sx_0) = s(tx_0) \in S(tx_0)$, or equivalently that $tx_0 \in S^{(-1)}(sx_0)$, contrary to assumption. Evidently $s \in (sx_0)X^{(-1)}$ so the admissible pair (F, E), obtained by an application of 3.18 to the point sx_0, is of index $(2,2)$. Since $s \in (sx_0)X^{(-1)}$ and $t \in S\backslash(sx_0)X^{(-1)}$, we have $\phi(s) \neq \phi(t)$, so the first part of the lemma follows. The isomorphism assertions are immediate in view of 3.18.

<u>Proof</u> of 3.13. We suppose now that (S, X) is a compact, totally disconnected, effective, trellis-like act such that $a \in Sa$ for each $a \in X$, and that (s,x) and (t,y) are distinct points of $S \times X$. We wish to prove that there is a homomorphism (ϕ,ψ) such that $(\phi(s), \psi(x)) \neq (\phi(t), \psi(y))$. There are two cases to consider: either $x \neq y$, or $x = y$ and $s \neq t$.

(I) $x \neq y$. We have $\Delta_X \subseteq X \times X\backslash\{(x,y)\} = V$ a proper open subset of $X \times X$ and if we choose any proper open subset U of $S \times S$, lemma 3.11 applies to produce a clopen admissible pair $(F_0, E_0) \leq (U, V)$, so that $(S/F_0, X/E_0) = (S_0, X_0)$ is a finite discrete and trellis-like act such that $a \in S_0a$ for each $a \in X_0$, and furthermore $\overline{x} \neq \overline{y}$ where in general $\overline{t}$ is the equivalence class in X_0 containing the element t of X. (The same notation will be used for the equivalence classes in S_0.) Now since $\overline{x} \neq \overline{y}$, as noticed

earlier the hypothesis of effectivity is not needed to deduce from 3.20 that there is an admissible pair (F, E) of index $(1,2)$ or $(2,2)$ on $(S_\circ, X_\circ)$ such that the equivalence classes in $X_\circ$ containing $\overline{x}$ and $\overline{y}$ are distinct. Now if (ϕ,ψ) is the completion of the following diagram, we have $\psi(x) \neq \psi(y)$, so case I follows.

$$\begin{tikzcd}
(S, X) \arrow[r, "\text{canonical}"] \arrow[rrd, dashed, "(\phi,\psi)"'] & (S_\circ, X_\circ) \arrow[r, "\text{canonical}"] & (S_\circ/F, X_\circ/E) \arrow[d, "\text{isomorphism}"] \\
 & & (\underline{2},\underline{2})
\end{tikzcd}$$

(Of course if the index of (F, E) is $(1,2)$ the isomorphism is to $(\underline{1},\underline{2})$ which is a subact of $(\underline{2},\underline{2})$; this inclusion is not shown on the diagram.)

(II) $x = y$ and $s \neq t$. Since the action of S on X is effective, there is some $w \in X$ such that $sw \neq tw$. We assume without loss of generality that $tw \notin S^{(-1)}(sw)$ and hence conclude as in the proof of 3.20 that $t \notin (sw)X^{(-1)}$. Now $(\Delta_S, \Delta_X) \leq (S \times S\backslash(s,t), X \times X\backslash(sw,tw))$ and since (S, X) is a compact and totally disconnected act we apply 3.11 to deduce the existence of a clopen admissible pair $(F_\circ, E_\circ) \leq (S \times S\backslash(s,t), X \times X\ (sw, tw))$ making $(S_\circ, X_\circ) = (S/F_\circ, X/E_\circ)$ a finite discrete act, which is trellis-like. Since $tw \notin S^{(-1)}(sw)$ and $sw \in S^{(-1)}sw$, we have $\overline{tw} \neq \overline{sw}$, and in this case the effectivity hypothesis in 3.20 is not needed to conclude that there is a clopen pair (F, E) on $(S_\circ, X_\circ)$ of index $(1,2)$ or $(2,2)$ such that $(\overline{sw},\overline{tw}) \notin E$. We claim that in addition $(\overline{s},\overline{t}) \notin F$. For F induces the decomposition

$\{S_\circ \backslash (\overline{sw})X_\circ^{(-1)},\ (\overline{sw})X_\circ^{(-1)}\}$ of $S_\circ$, and since we have already observed that $\overline{t} \in S_\circ \backslash (\overline{sw})X_\circ^{(-1)}$, we need only see that $\overline{s} \in \overline{sw}X_\circ^{(-1)}$; but this is equivalent to saying $\overline{sw} \in \overline{s}X_\circ$ which is certainly the case. Hence F has index 2 and, letting (ϕ,ψ) be the completion of the following diagram, we have proved $\phi(s) \neq \phi(t)$; hence case II follows.

$$\begin{tikzcd}
(S, X) \arrow[r, "\text{canonical}"] \arrow[drr, dashed, "(\phi,\psi)"'] & (S_\circ, X_\circ) \arrow[r, "\text{canonical}"] & (S_\circ/F, X_\circ/E) \arrow[d, "\text{isomorphism}"] \\
& & (\underline{2},\underline{2})
\end{tikzcd}$$

To conclude the proof let $(P, Q) = \prod_{(s,x)\neq(t,y)} (\underline{2},\underline{2})$. Since for each copy of $(\underline{2},\underline{2})$ there is a homomorphism (ϕ,ψ): $(S, X) \to (2,2)$ by the above constructions, then if we apply 3.3 we will obtain a homomorphism $(\overline{\Phi}, \overline{\Psi})$: $(S, X) \to (P, Q)$. Now if x and y are distinct points of X it follows from (I) that $\overline{\Psi}(x) \neq \overline{\Psi}(y)$ so that $\overline{\Psi}$ is 1-1; similarly $\overline{\Phi}$ is 1-1, so that $(\overline{\Phi}, \overline{\Psi})$ is an isomorphism into (P, Q) and 3.13 is proved.

There is an easy corollary to 3.13 which is almost immediate.

<u>Lemma 3.21</u>. If (S, X) is such an idempotent act that $X = Sz$ for some $z \in X$, then $x \in Sx$ for each $x \in X$.

<u>Proof</u>. For if $x \in X$ then for some $s \in S$, $x = sz = s(sz) = sx \in Sx$.

<u>Lemma 3.22</u>. If (S, X) is trellis-like and satisfies $X = Sz$ for some $z \in X$ and (F, E) is an admissible pair of index (1,2) or (2,2) then necessarily the index of (F, E) is (2,2).

Proof. If to the contrary $F = S \times S$ then suppose $(x,y) \in X \times X$. By hypothesis there is some $(t_1,t_2) \in S \times S$ such that $(x,y) = (t_1z,t_2z) = (t_1,t_2)(z,z) \in FE \subseteq E$, implying that the index of E is 1, which is absurd.

Corollary 3.23. If (S, X) is a compact, totally disconnected, effective, trellis-like act such that $X = Sz$ and neither S nor X has cardinal number 1, then (S, X) may be isomorphically embedded in a product of acts $(\underline{2},\underline{2})$ such that each factor is the homomorphic image of (S, X).

Proof. In view of 3.21, the hypotheses of 3.13 are all satisfied; if we now note that each admissible pair (F, E) constructed in the proof of 3.13 must have index (2,2) by 3.22, and that the separating homomorphism (ϕ,ψ): $(S, X) \to (\underline{2},\underline{2})$ is surjective on both coordinates in each case, the corollary follows.

APPENDIX

Actions of Order (2,2)

The <u>order</u> of an act (S, X) is a pair (p,q) of cardinal numbers such that p = card S and q = card X.

Let S = {0,1} and X = {a,b} be considered just as sets. It is well-known [19] that there are only four non-isomorphic semigroups on the set S; we denote these as S_1, S_2, S_3, S_4 and give their Cayley tables below.

S_1

	0	1
0	0	1
1	0	1

S_2

	0	1
0	0	0
1	0	0

S_3

	0	1
0	0	1
1	1	0

S_4

	0	1
0	0	0
1	0	1

In this appendix we summarize the computation of all actions of S_i upon X for each i, i = 1, 2, 3, 4. The actions are conveniently given by their Cayley tables where tx appears as the entry in the t-row and x-column of the table. The sixteen functions $S \times X \xrightarrow{\mu_j} X, 1 \leq j \leq 16$ are given below.

(1)

	a	b
0	a	a
1	a	a

(2)

a	a
a	b

(3)

a	a
b	b

(4)

a	b
a	b

$\begin{array}{\|cc}\hline b & a \\ a & b\end{array}$	$\begin{array}{\|cc}\hline a & a \\ b & a\end{array}$	$\begin{array}{\|cc}\hline b & a \\ b & a\end{array}$	$\begin{array}{\|cc}\hline a & b \\ b & a\end{array}$
(5)	(6)	(7)	(8)
$\begin{array}{\|cc}\hline a & b \\ a & a\end{array}$	$\begin{array}{\|cc}\hline b & b \\ a & a\end{array}$	$\begin{array}{\|cc}\hline b & a \\ a & a\end{array}$	$\begin{array}{\|cc}\hline a & b \\ b & b\end{array}$
(9)	(10)	(11)	(12)
$\begin{array}{\|cc}\hline b & a \\ b & b\end{array}$	$\begin{array}{\|cc}\hline b & b \\ b & a\end{array}$	$\begin{array}{\|cc}\hline b & b \\ a & b\end{array}$	$\begin{array}{\|cc}\hline b & b \\ b & b\end{array}$
(13)	(14)	(15)	(16)

In several cases, there is an obvious one-to-one correspondence between some S_i and X which makes X into a semigroup isomorphic to S in such a way that the action of S_i upon X is just semigroup multiplication; we will refer to this phenomenon by saying the action <u>copies</u> the semigroup's multiplication. For convenience we say S_i acts <u>constantly</u> on X if $tx = x_0$ for all t,x and some fixed $x_0 \in X_i$. S_i acts <u>projectively</u> if tx = x for all t,x. A <u>trivial</u> action is either constant or projective.

In the following summary, (S_i, μ_j) denotes the act (S_i, X, μ_j).

(I) Trivial actions

(S_i, μ_j) for all i and j = 1, 4, 16.

(S_1, μ_4), (S_2, μ_1), (S_2, μ_{16}) copy multiplication.

(II) Non-trivial actions

(S_3, μ_8) copies multiplication

(S_4, μ_2) and (S_4, μ_{15}) copy multiplication and are isomorphic.

For each remaining j, we exhibit an $s,t \in S_i$ and an $x \in X$ such that $(st)x \neq s(tx)$. We write $s \cdot x = \mu_j(s,x)$.

μ_2: In S_1, S_3, $0(1\cdot b) \neq (0\cdot 1)b$; In S_2, $1(1\cdot b) \neq (1\cdot 1)b$

μ_3: In S_1, S_2, S_4, $1(0\cdot a) \neq (1\cdot 0)a$; In S_3, $1(1\cdot a) \neq (1\cdot 1)a$

μ_5: In S_1, S_2, S_3, S_4, $0(0\cdot a) \neq (0\cdot 0)a$

μ_6: In S_1, S_4, $1(1\cdot a) \neq (1\cdot 1)a$; In S_2, $1(0\cdot a) \neq (1\cdot 0)a$; In S_3, $1(1\cdot b) \neq (1\cdot 1)b$

μ_7: In S_1, S_2, S_3, S_4, $0(0\cdot a) \neq (0\cdot 0)a$

μ_8: In S_1, S_4, $1(1\cdot a) \neq (1\cdot 1)a$; In S_2, $1(0\cdot a) \neq (1\cdot 0)a$

μ_9: In S_1, S_2, S_4, $1(0\cdot b) \neq (1\cdot 0)b$; In S_3, $0(1\cdot b) \neq (0\cdot 1)b$

μ_{10}: In S_1, S_2, S_4, $1(0\cdot a) \neq (1\cdot 0)a$; In S_3 $0(1\cdot a) \neq (0\cdot 1)a$

μ_{11}: In S_1, S_2, S_3, S_4, $0(0\cdot a) \neq (0\cdot 0)a$

μ_{12}: In S_1, S_2, S_4, $1(0\cdot a) \neq (1\cdot 0)a$; In S_3, $1(1\cdot a) \neq (1\cdot 1)a$

μ_{13}: In S_1, S_2, S_3, S_4, $0(0\cdot a) \neq (0\cdot 0)a$

μ_{14}: In S_1, S_4, $1(1\cdot b) \neq (1\cdot 1)b$; In S_2, S_3, $1(1\cdot a) \neq (1\cdot 1)a$

μ_{15}: In S_1, $0(1\cdot a) \neq (0\cdot 1)a$; In S_2, S_3, $1(1\cdot a) \neq (1\cdot 1)a$.

BIBLIOGRAPHY

[1] Aczél, J. and Wallace, A. D. A Note on Generalizations of Transitive Systems of Transformations, Colloq. Math (XVII) Fasc I 1967 pp. 29-34.

[2] Bednarek, A. R. and Norris, E. M. Congruences and Ideals in Machines (Submitted).

[3] Bednarek, A. R. and Wallace, A. D. Relative Ideals and Their Complements, Rev. Rom. de Math. Pures et App. (XI) 1966 pp. 13-22.

[4] ______________, Equivalences on Machine State Spaces, Mathematický Časopis (17) 1967 pp. 3-9.

[5] ______________, Finite Approximants of Compact Totally Disconnected Machines, Math. Syst. J. (I) 1967 pp. 209-216.

[6] Clifford, A. H. and Preston, G. B. Algebraic Theory of Semigroups, Mathematical Surveys Number 7, American Math. Soc. 1961.

[7] Cohn, P. M. Universal Algebra, Harper and Row, New York 1965.

[8] Day, J. M. and Wallace, A. D. Semigroups Acting on Continua, J. Aust. Math Soc. (VII) 1967 pp.327-340.

[9] ______________, Multiplication Induced in the State Space of an Act, Math. Syst. J. (I) 1967 pp. 305-314.

[10] Eilenberg, S. and Steenrod, N. Foundations of Algebraic Topology, Princeton University Press, Princeton, N. J. 1952.

[11] Freyd, P. Abelian Categories, Harper and Row, New York 1964.

[12] Gottschalk, W. H. and Hedlund, G. A. Topological Dynamics, AMS Coll. Pub., Vol. 36, Providence, 1955.

[13] Hoffman, K. H. and Mostert, P. S. Elements of Compact Semigroups, Charles E. Merrill Books, Columbus, Ohio 1966.

[14] Koch, R. J. and Wallace, A. D. Maximal Ideals in Compact Semigroups, Duke Math. J. (21) 1954 pp. 681-685.

[15] Mitchell, B. Theory of Categories, Academic Press, New York 1965.

[16] Numakura, K. Theorems on Compact Totally Disconnected Semigroups and Lattices, PAMS (8) 1957 pp. 623-626.

[17] Paalman-de Miranda, A. B. Topological Semigroups, Mathematisch Centrum, Amsterdam 1964.

[18] Rees, D. On Semigroups, Proc. Camb. Phil. Soc. (36) 1940 pp. 387-400.

[19] Tetsuya, H., et. al. All Semigroups of Order at Most 5, J. of Gakugei Tokushimu University (VI), December 1955 pp. 19-39 plus an unpaginated errata sheet.

[20] Wallace, A. D. Retractions in Semigroups, Pac. J. Math. (7) 1957 pp.1413-1417.

[21] _______________, Relative Ideals in Semigroups I, Colloq. Math. (9) 1962 pp.55-61.

[22] _______________, Relative Ideals in Semigroups II, Acta. Math. Acad. Sci. Hung. (XIV) 1963 pp. 137-148.

[23] _______________, Recursions with Semigroup State Spaces, Rev. Rom. de Math. Pures et App. (XII) 1967 pp. 1411-1415.

BIOGRAPHICAL SKETCH

Eugene Michael Norris was born July 4, 1938, in New York City. In June, 1956, he was graduated from Hillsborough High School in Tampa, Florida. A one-semester residence at the University of Tampa followed, to be succeeded by a two-year stint as an electronics technician on the Atlantic Missile Range. During 1959 he attended the University of Florida and then returned to his position as electronics technician on Ascension Island and Grand Bahama Island. From March 1961 until September 1962 he was employed by the National Aeronautics and Space Administration at Goddard Space Flight Center. He entered the University of South Florida in September 1962 and received the degree of Bachelor of Arts in mathematics in August 1964. Since that time he has been in the department of mathematics at the University of Florida as a graduate student, having spent the academic year 1965-1966, as well as the present year, as an interim instructor.

Eugene Michael Norris is married to the former Lois McGibney. He is a member of the Mathematical Association of America and of the American Mathematical Society.

This dissertation was prepared under the direction of the chairman of the candidate's supervisory committee and has been approved by all members of that committee. It was submitted to the Dean of the College of Arts and Sciences and to the Graduate Council, and was approved as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

June, 1969

E. Ruffin Jones

Dean, College of Arts and Sciences

Dean, Graduate School

Supervisory Committee:

A. R. Bednarek

Chairman

D. P. Stadtlander

K. N. Sigmon

A. S. Schumacher